Numerical Stability of the Euler scheme for BSDEs

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Outline

Backward Sto. Diff. Eq.

Backward SDE on $[0, T]$:

$$Y_t = \xi + \int_t^T f(Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s$$

$\hookrightarrow \xi \in L^2(\mathcal{F}_T)$, f is a Lipschitz function,
$\hookrightarrow (Y, Z)$ adapted solution.

Non-linearity and finance

Example

The stock price is given by

$$X_t = X_0 + \int_0^t \mu X_s \mathrm{d}s + \int_0^t \sigma X_s \mathrm{d}W_s$$

The price of an european option with payoff g, assuming different rate for borrowing (R) and lending (r) is given by

$$Y_t = g(X_T) + \int_t^T \left(-rY_s + \frac{\mu - r}{\sigma} Z_s + (R-r)[Y_s - \frac{Z_s}{\sigma}]_-\right)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s$$

$\hookrightarrow$ These are the dynamics of the value of the optimal hedging portfolio.

- Non-linearity coming from $f(y,z) = -ry + \frac{\mu - r}{\sigma} z + (R-r)[y - \frac{z}{\sigma}]_-$

Deriving the scheme

We are given an equidistant grid $\pi = \{0 = t_0 < ... < t_i < ... < t_n = T\}$, define $h = T/n$.

- Start with: $Y_{t_i} + \int_{t_i}^{t_{i+1}} Z_s \mathrm{d}W_s = Y_{t_{i+1}} + \int_{t_i}^{t_{i+1}} f(Y_s, Z_s)\mathrm{d}s \quad (1)$

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$$\hookrightarrow \quad Z_i := \mathbb{E}_{t_i}[H_i Y_{i+1}] \quad \text{with} \quad H_i := h^{-1}\Delta W_i\ .$$

Euler Scheme

- The Scheme: given the terminal condition $Y_n = \xi$, the transition from step $i+1$ to i is

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- Goals: Understand the qualitative behaviour of the scheme in practice.

ODEs and BSDEs

- Things can go wrong already for ODEs: $y' = f(y)$ with $f(y) = -ay$, $a > 0$.
  Explicit Euler scheme satisfies: $y_n = (1 - ah)^n y_0$
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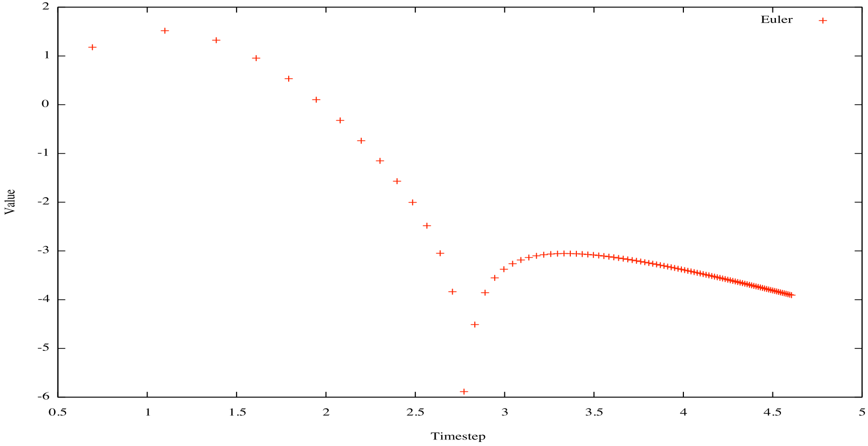
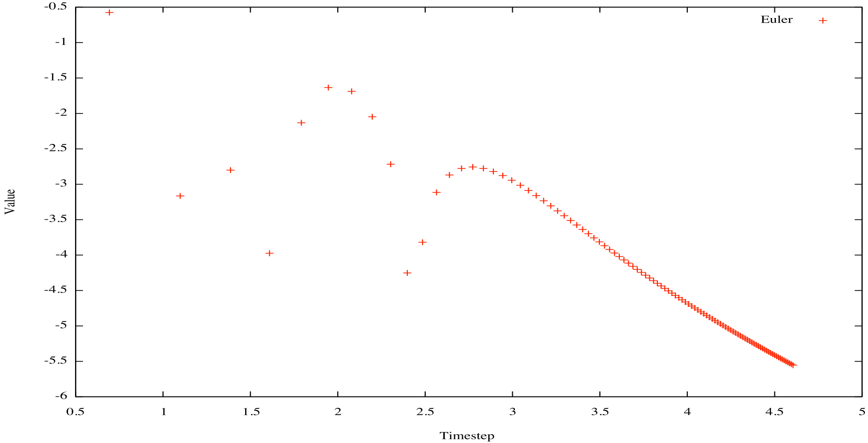
  $\bar{h} = \frac{2}{a}$ is a critical value.
- What happens in the 'pure' BSDEs setting?
  $\hookrightarrow$ We consider $f(Z) = bZ$ and $dim(Y) = dim(Z) = 1$.
  $\hookrightarrow$ The terminal condition is given by $\cos(\widehat{W}_T)$.
  $\hookrightarrow$ $\widehat{W}$ is a (recombining) trinomial tree for the brownian motion W.

$f(Z) = bZ,\ b = 5,\ T = 1$



$f(Z) = bZ$, $b = 1$, $T = 10$


-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
-4.5
-5
-5.5
-6
Value
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
Timestep
Euler

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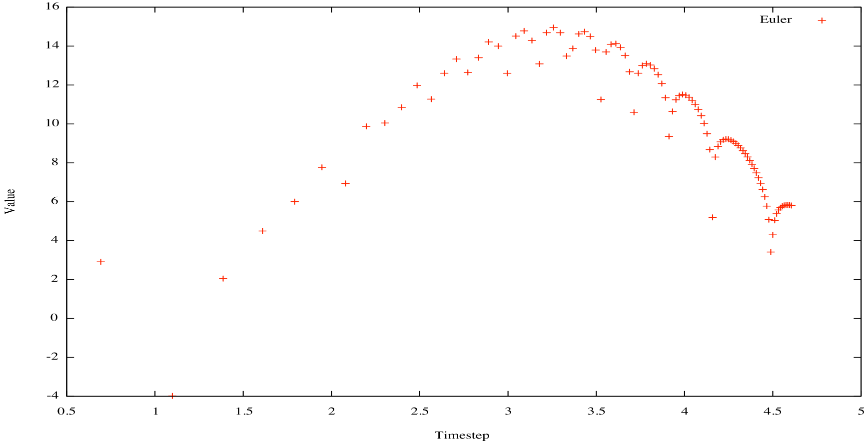
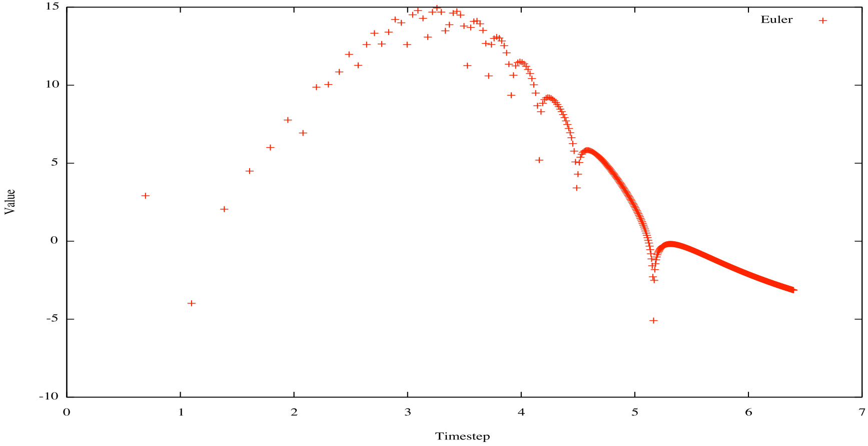


$f(Z) = bZ$, $b = 5$, $T = 10$, a lot of steps



Outline

Framework

We will discuss the two following schemes:

- BTZ-scheme 'Implicit' Euler:

$$Y_i := \mathbb{E}_{t_i}[Y_{i+1} + hf(Y_i, Z_i)]$$
$$Z_i := \mathbb{E}_{t_i}[H_i Y_{i+1}]$$

- 'Explicit' Euler:

$$Y_i := \mathbb{E}_{t_i}[Y_{i+1} + hf(Y_{i+1}, Z_i)]$$
$$Z_i := \mathbb{E}_{t_i}[H_i Y_{i+1}]$$

- $\mathbb{E}_{t_i}[H_i] = 0$, $\mathbb{E}_{t_i}\left[|H_i|^2\right] \leq \Lambda$, for some given $\Lambda > 0$.

Remarks

The analysis covers various types of schemes:

- Theoretical ones given in the introduction and $H_i := \frac{1}{h}(W_{t_{i+1}} - W_{t_i})$.
- Numerical scheme using trees e.g.
  - (i) Trinomial:

$$\mathbb{P}(H_i = \pm \frac{3}{\sqrt{h}}) = \frac{1}{6}\,,\ \mathbb{P}(H_i = 0) = \frac{2}{3}.$$

  - (ii) Binomial:

$$\mathbb{P}(H_i = \pm \frac{1}{\sqrt{h}}) = \frac{1}{2}.$$

$\hookrightarrow$ H_i is bounded.

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- $f(0,0) = 0 \implies (0,0)$ is solution with terminal condition 0.

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$$|f(y,z) - f(y',z)| \leq L^Y |y - y'| \quad \textbf{(Lip y)}$$

and/or

$$yf(y,0) \leq -l^Y |y|^2 \quad \textbf{(Mon)}$$

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where L^Y, l^Y are non-negative real numbers.

- We also assume that f is Lipschitz continuous in z (uniformly) i.e.

$$|f(y,z) - f(y,z')| \leq L^Z |z - z'| \; . \quad \textbf{(Lip z)}$$

Behaviour of the true solution

Question: can we obtain sometimes a uniform bound (in T) for Y ?

In our setting (Lip z + monotone y), if

- in the multidimensional case (for Y): $(L^Z)^2 \leq 2l^Y$, $\|\xi\|_\infty < \infty$
- in the one-dimensional case (for Y): simply $\|\xi\|_\infty < \infty$

then

$$|Y_t| \leq \|\xi\|_\infty .$$

(remark: for all T.)

Some Definitions

Let ξ be a bounded terminal condition (random).

- Numerical stability: We say that the scheme is numerically stable if there exists $h^* > 0$, such that for all $h \leq h^*$

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$\hookrightarrow$ In practice, we could expect 2 regimes for the scheme:

- $h < \bar{h}$: scheme returns a 'reasonable' value
- $h > \bar{h}$: scheme is unstable

Strict monotony ($l^Y > 0$) and multidimensional Y

Theorem

Assume that

$$1 - \frac{\left(L^Y\sqrt{h^*} + L^Z\sqrt{\Lambda}\right)^2}{2l^Y} \geq 0 \tag{1}$$

then the pseudo-explicit scheme is numerically stable for the Y part.
Assume that

$$\frac{1}{\Lambda} - \frac{|L^Z|^2}{2l^y} \geq 0\,, \tag{2}$$

then the scheme is A-stable.

Strict monotony ($l^Y > 0$) and one dimensional Y

Theorem

Assume that

$$1 - h^* \frac{|L^Y|^2}{2l^Y} - L^Z h^* \max_i |H_i| \geq 0 \tag{3}$$

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$$1 - L^Z h^* \max_i |H_i| \geq 0 \tag{4}$$

then the implicite scheme is numerically stable.

Sketch of proof (dimension 1)

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- If $(1 + h\gamma_i H_i) \geq 0$,

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- Comparison Theorem in this case.

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Von Neumann Stability Analysis: $f(y,z) = -ay + bz$

- $dim(Y) = 1$, Scheme given by *time discretization* only (i.e. $H_i = h^{-1}(W_{t_{i+1}} - W_{t_i})$ unbounded!).

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- Necessary condition for numerical stability.

Von Neumann Stability Analysis

Theorem

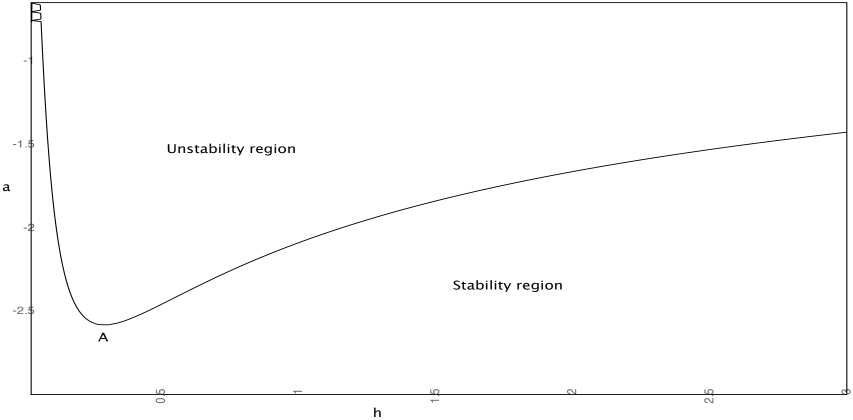
- *Implicit scheme is VN stable if* $b^2h \leq 1$ *or* $b^2h > 1$ *and*

$$(1 + ah) - b^2 h e^{\frac{1}{b^2 h} - 1} \geq 0$$

- *Pseudo-explicit scheme is VN stable if* $b^2h \leq (1 + ah)^2$ *and* $h \leq -2/a$ *or* $b^2h > (1 + ah)^2$ *and*

$$1 - b^2 h e^{\frac{(1+ah)^2}{b^2 h} - 1} \geq 0.$$

VN stability region - Implicit Scheme - b=5



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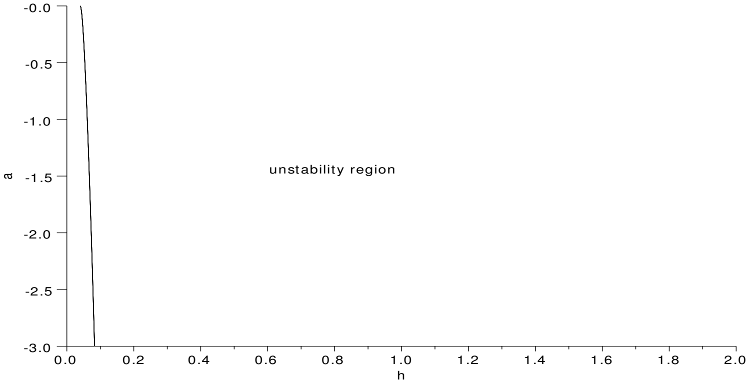
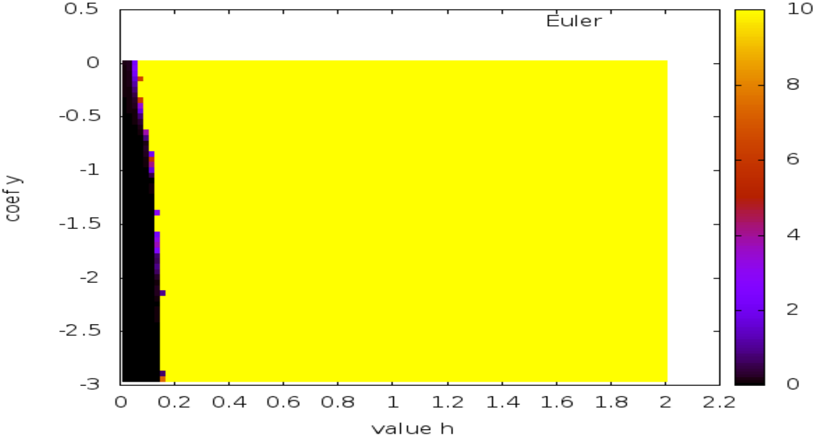
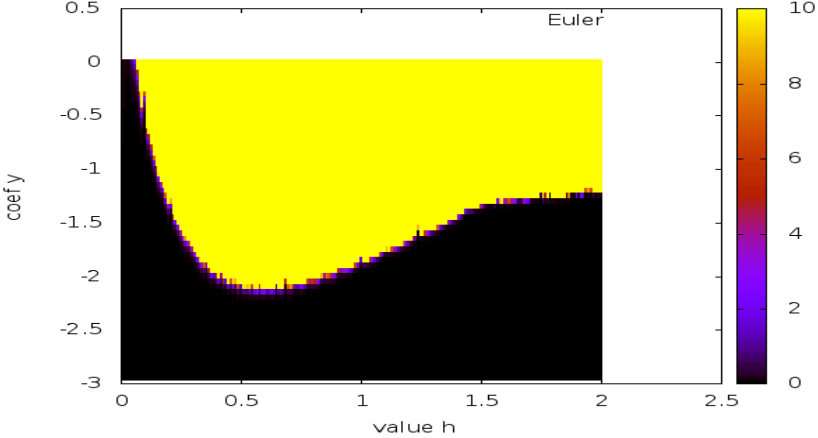


Numerical illustration $f(y,z) = -ay + bz$, $b = 5$

Yellow = unstable, correct Y_0 value $\simeq 0$.



Numerical illustration $-ay + bz$, $b = 5$



VN Stability - Empirical Scheme - Analysis, $f(y,z) = bz$

- For $k \in \mathbb{R}$, $\xi = e^{\mathbf{i}k\widehat{W}_{t_n}}$ (binomial tree),

$$Y_i = y_i e^{\mathbf{i}k\widehat{W}_{t_i}}$$

with

$$y_i = \lambda y_{i+1} \quad \text{with} \quad \lambda := \mathbb{E}\Big[(1 + b\Delta\widehat{W}_i)e^{\mathbf{i}k\Delta\widehat{W}_i}\Big]$$

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- The scheme is VN stable iff $|\lambda| \leq 1$ i.e. (after some computations)

$$h|b|^2 \leq 1 \ !$$

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- Remark: observe that the dimension of b (so W) impacts the stability of the scheme.

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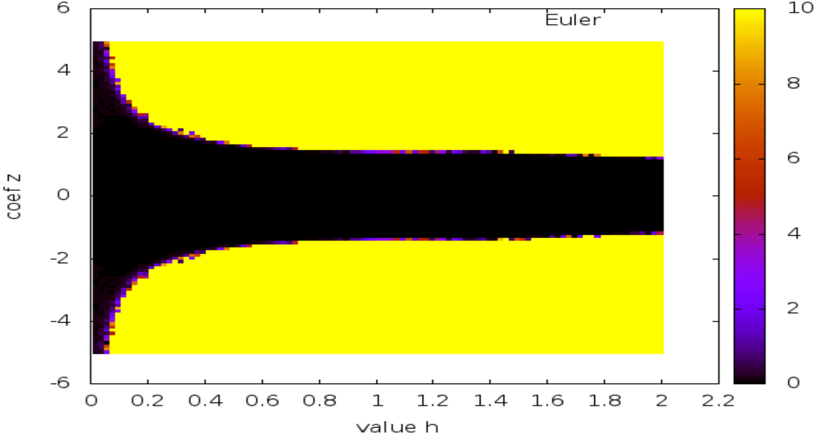
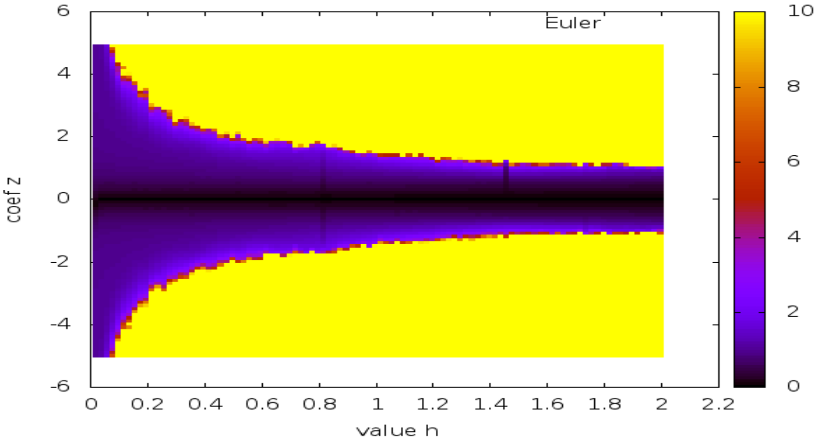


Numerical illustration $f(z) = b|z|$



Numerical illustration $f(z) = sin(bz)$