Automaton groups with unsolvable conjugacy problem

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(Joint work Z. Sunic)

Outline

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Main result

Theorem (Sunic-V.)

There exist automaton groups with unsolvable conjugacy problem.

It is a direct consequence of...

Theorem (Sunic-V.)

Let $\Gamma \leqslant GL_d(\mathbb{Z})$ be f.g. Then, $\mathbb{Z}^d \rtimes \Gamma$ is an automaton group.

Theorem (Bogopolski-Martino-V.)

There exists $\Gamma \leqslant GL_d(\mathbb{Z})$ f.g. such that $\mathbb{Z}^d \rtimes \Gamma$ has unsolvable conjugacy problem.

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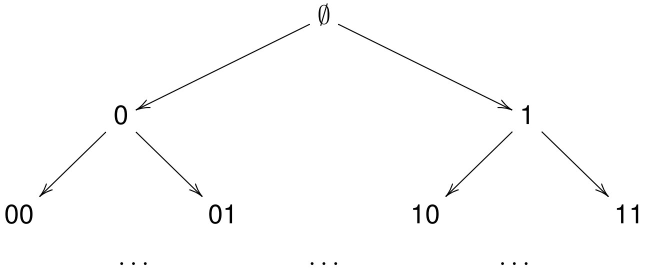
Tree automorphisms

Let X be an alphabet on k letters, and let X^* be the free monoid on X, thought as a rooted k-ary tree:



Definition

- Every *tree automorphism* g decomposes as a *root permutation* $\pi_g : X \to X$, and k *sections* $g|_x$, for $x \in X$:

$$g(xw) = \pi_g(x) g|_x(w).$$

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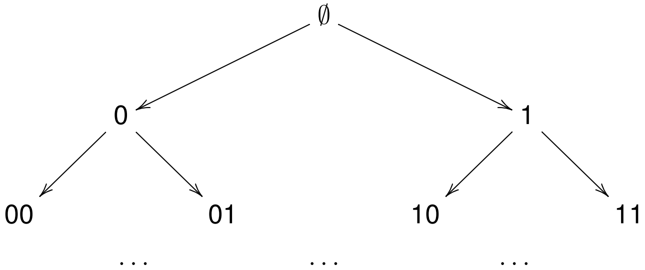

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Automaton groups

Definition

- *A set of tree automorphisms is self-similar if it contains all sections of all of its elements.*
- *The group $G(\mathcal{A})$ of tree automorphisms generated by a finite self-similar set $\mathcal{A}$ is called an automaton group.*

The Grigorchuk group: $G = \langle 1, \alpha, \beta, \gamma, \delta \rangle$, where

$$\alpha = \sigma(1,1), \quad \beta = 1(\alpha,\gamma), \quad \gamma = 1(\alpha,\delta), \quad \delta = 1(1,\beta).$$

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Affinities of p-adic integers

Definition

Let $\mathcal{M} = \{M_1, \ldots, M_m\}$ be integral $d \times d$ matrices with non-zero determinants. Let $p \geqslant 2$ be a prime not dividing any of these determinants (thus, M_i is invertible over the ring $\mathbb{Z}_p$ of p-adic integers).

For an integral $d \times d$ matrix M and $\mathbf{v} \in \mathbb{Z}^d$, consider the invertible affine transformation ${}_{\mathbf{v}}M \colon \mathbb{Z}_p^d \to \mathbb{Z}_p^d$, ${}_{\mathbf{v}}M(\mathbf{u}) = \mathbf{v} + M\mathbf{u}$.

Let

$$G_{\mathcal{M},p} = \langle \{ {}_{\mathbf{v}}M \mid M \in \mathcal{M},\ \mathbf{v} \in \mathbb{Z}^d \} \rangle \leqslant Aff_d(\mathbb{Z}_p).$$

Lemma

If, in addition, $\det M_i = \pm 1$, then $G_{\mathcal{M},p} \cong \mathbb{Z}^d \rtimes \Gamma$, where $\Gamma = \langle M_1, \ldots, M_m \rangle \leqslant \mathrm{GL}_d(\mathbb{Z})$.

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Since ${}_{\mathbf{v}}M = \tau_{\mathbf{v}}\, {}_{\mathbf{0}}M$, *we have* $G_{\mathcal{M},p}$ *generated by* ${}_{\mathbf{0}}M$ *for* $M \in \mathcal{M}$, *and* $\tau_{\mathbf{e}_i}$, *where the* $\mathbf{e}_i$*'s are the canonical vectors.*

If $M \in \mathrm{GL}_d(\mathbb{Z})$, *then* ${}_{\mathbf{v}}M \in \mathit{Aff}_d(\mathbb{Z}_p)$ *restricts to an integral bijective affine transformation* ${}_{\mathbf{v}}M \in \mathit{Aff}_d(\mathbb{Z})$; *hence, we can view* $G_{\mathcal{M},p} \leqslant \mathit{Aff}_d(\mathbb{Z})$ *(and is independent from p; let's denote it by* $G_{\mathcal{M}}$*).*

They get multiplied as

$$ {}_{\mathbf{v}}M_{\mathbf{v}'}M' : \mathbf{u} \longrightarrow \mathbf{v}' + M'\mathbf{u} \longrightarrow \begin{array}{l} \mathbf{v} + M(\mathbf{v}' + M'\mathbf{u}) = \\ (\mathbf{v} + M\mathbf{v}') + MM'\mathbf{u} = \\ {}_{\mathbf{v}+M\mathbf{v}'}(MM')(\mathbf{u}). \end{array} $$

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$$\det M_i = \pm 1 \quad \Rightarrow \quad G_{\mathcal{M}} \cong \mathbb{Z}^d \rtimes \Gamma,$$

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Definition

Elements in $\mathbb{Z}_p$ may be (uniquely) represented as right infinite words over $Y_p = \{0, \ldots, p-1\}$:

$$y_1 y_2 y_3 \cdots \quad \longleftrightarrow \quad y_1 + p \cdot y_2 + p^2 \cdot y_3 + \cdots .$$

Similarly, elements of $\mathbb{Z}_p^d$ (the free d-dimensional module, viewed as column vectors), may be (uniquely) represented as right infinite words over $X_p = Y_p^d = \{(y_1, \ldots, y_d)^T \mid y_i \in Y_p\}$:

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Note that $|Y_p| = p$ and $|X_p| = p^d$.

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For $\mathbf{v} \in \mathbb{Z}^d$*, define vectors* $\mathrm{Mod}(\mathbf{v}) \in X_p$ *and* $\mathrm{Div}(\mathbf{v}) \in \mathbb{Z}^d$ *s.t.* $\mathbf{v} = \mathrm{Mod}(\mathbf{v}) + p \cdot \mathrm{Div}(\mathbf{v})$.

Lemma

For every $\mathbf{v} \in \mathbb{Z}^d$, $M \in Mat_d(\mathbb{Z})$, *and* $\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3 \cdots \in \mathbb{Z}_p^d$, *we have*

$$_{\mathbf{v}}M(\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\cdots) = \mathrm{Mod}(\mathbf{v}+M\mathbf{x}_1) + p \cdot_{\mathrm{Div}(\mathbf{v}+M\mathbf{x}_1)} M(\mathbf{x}_2\mathbf{x}_3\mathbf{x}_4\cdots).$$

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For $M \in \mathcal{M}$, let V_M be the set of integral vectors with coordinates between $-\|M\|$ and $\|M\| - 1$ (note that $|V_M| = (2\|M\|)^d$).

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Construct the automaton $\mathcal{A}_{M,p}$:

- *Alphabet: X_p.*
- *States: $m_{\mathbf{v}}$ for $\mathbf{v} \in V_M$, with root permutation and sections*

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- *States: $m_{\mathbf{v}}$ for $\mathbf{v} \in V_M$, with root permutation and sections*

$$m_{\mathbf{v}}(\mathbf{x}) = \mathrm{Mod}(\mathbf{v} + M\mathbf{x}), \quad \text{and} \quad m_{\mathbf{v}}|_{\mathbf{x}} = m_{\mathrm{Div}(\mathbf{v}+M\mathbf{x})}.$$

- *Straightforward to see that sections are again states.*

$G_{\mathcal{M}}$ is an automaton group

$$_{\mathbf{v}}M(\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\cdots) = \mathrm{Mod}(\mathbf{v}+M\mathbf{x}_1) + p\cdot_{\mathrm{Div}(\mathbf{v}+M\mathbf{x}_1)} M(\mathbf{x}_2\mathbf{x}_3\mathbf{x}_4\cdots).$$

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For $M \in \mathcal{M}$, let V_M be the set of integral vectors with coordinates between $-\|M\|$ and $\|M\|-1$ (note that $|V_M| = (2\|M\|)^d$).

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The state $m_{\mathbf{v}} \in \mathcal{A}_{M,p}$ acts on a vector $\mathbf{u} = \mathbf{x}_1\mathbf{x}_2\mathbf{x}_3 \cdots \in \mathbb{Z}_p^d$ as $m_{\mathbf{v}}(\mathbf{u}) = {}_{\mathbf{v}}M(\mathbf{u})$.

Definition

Construct the automaton $\mathcal{A}_{\mathcal{M},p}$ as the disjoint union of the automata $\mathcal{A}_{M_1,p}, \ldots, \mathcal{A}_{M_m,p}$.

- *Alphabet: X_p,*
- *It has $2^d \sum_{i=1}^{m} ||M_i||^d$ states.*

Proposition

$G_{\mathcal{M},p}$ is an automaton group generated by the automaton $\mathcal{A}_{\mathcal{M},p}$ (over an alphabet of size p^d, and having $2^d \sum_{i=1}^{m} ||M_i||^d$ states).

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Outline

Orbit decidability

Theorem (Bogopolski-Martino-V.)

There exists $\Gamma \leqslant \mathrm{GL}_d(\mathbb{Z})$ f.g. such that $\mathbb{Z}^d \rtimes \Gamma$ has unsolvable conjugacy problem.

Definition

Let G be a f.g. group. A subgroup $\Gamma \leqslant \mathrm{Aut}(G)$ is said to be orbit decidable (O.D.) if there is an algorithm to decide, given $u, v \in G$, whether $v = \alpha(u)$ for some $\alpha \in \Gamma$.

Observation (folklore)

The full group $\mathrm{Aut}(\mathbb{Z}^d) = \mathrm{GL}_d(\mathbb{Z})$ is orbit decidable.

Proof. *For $u, v \in \mathbb{Z}^d$, there exists $A \in \mathrm{GL}_d(\mathbb{Z})$ such that $v = Au$ if and only if $gcd(u_1, \ldots, u_d) = gcd(v_1, \ldots, v_d)$.*

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Every finitely index subgroup of $GL_d(\mathbb{Z})$ is O.D.

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Connection to semidirect products

Observation (B-M-V)

Let H be f.g., and $\Gamma \leqslant \mathrm{Aut}(H)$ f.g. If $H \rtimes \Gamma$ has solvable CP, then $\Gamma \leqslant \mathrm{Aut}(H)$ is orbit decidable.

Proof. $G = H \rtimes \Gamma$ contains elements $(h, \gamma) \in H \times \Gamma$ operated like

$$(h_1, \gamma_1) \cdot (h_2, \gamma_2) = (h_1 \gamma_1(h_2), \gamma_1 \gamma_2)$$

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For $h_1, h_2 \in H \leqslant G$, we have $h_1 \sim_G h_2 \;\Leftrightarrow\; \exists (h, \gamma) \in H \rtimes \Gamma$ s.t.

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Hence, $h_1 \sim_G h_2 \;\Leftrightarrow\; \exists \gamma \in \Gamma$ and $h \in H$ s.t. $h_1 = h\gamma(h_2)h^{-1}$. □

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Proposition (Bogopolski-Martino-V. 2008)

Let G be a group, and let $A \leqslant B \leqslant \mathrm{Aut}(G)$ and $v \in G$ be such that $B \cap Stab(v) = 1$. Then,

$$OD(A) \text{ solvable} \quad \Rightarrow \quad MP(A, B) \text{ solvable.}$$

Proof. *Given $\varphi \in B \leq \mathrm{Aut}(G)$, let $w = v\varphi$ and*

$$\{\phi \in B \mid v\phi = w\} = B \cap (Stab(v) \cdot \varphi) = (B \cap Stab(v)) \cdot \varphi = \{\varphi\}.$$

So, deciding whether v can be mapped to w by somebody in A, is the same as deciding whether φ belongs to A. Hence,

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Orbit undecidable subgroups

Proposition (Bogopolski-Martino-V., 08)

For $d \geqslant 4$, there exist f.g., orbit undecidable, subgroups $\Gamma \leqslant \mathrm{GL}_d(\mathbb{Z})$.

Proof.

- *Take a copy of $F_2 = \langle P, Q \rangle$ inside $GL_2(\mathbb{Z})$.*
- *Take $F_2 \times F_2 \simeq B \leqslant GL_4(\mathbb{Z})$.*
- *The existence of v is technical.*
- *Take $A \leq B \simeq F_2 \times F_2$ with unsolvable membership problem.*
- *By previous Proposition, $A \leqslant \mathrm{GL}_4(\mathbb{Z})$ is orbit undecidable.*
- *Similarly for $A \leqslant \mathrm{GL}_d(\mathbb{Z})$, $d \geqslant 4$.* □

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Does there exist an orbit undecidable subgroup of $GL_3(\mathbb{Z})$?

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