Membership in the BNS invariant

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(Joint work with D. Kharobaei, J. Delgado and B. Cavallo)

Outline

Definition

Let $\mathcal{G}$ be the class of f.p. groups. We are interested in algorithmic recognition of subclasses $\mathcal{H} \subseteq \mathcal{G}$:

- *Membership: given $G \in \mathcal{G}$, decide whether it belongs to $\mathcal{H}$ or not.*
- *Isomorphism: given $H_1, H_2 \in \mathcal{H}$, decide whether $H_1 \simeq H_2$.*
- *Good presentations: given $H \in \mathcal{H}$, find a "good" pres. for H.*

Many of these problems are algorithmically unsolvable:

- Triviality: membership in $\mathcal{H} = \{1\}$;
- Freeness: membership in $\mathcal{F} = \{\text{f.g. free groups}\}$;
- Isomorphism: in $\mathcal{G}$ and in many classes $\mathcal{H}$;

But there are also positive results for some classes $\mathcal{H}$...

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$\mathbb{Z}$-extensions

Definition

Let $H = \langle X \mid R \rangle$ be a group and $\alpha \in Aut(H)$. The semidirect extension of H given by α is:

$$H_\alpha = H \rtimes_\alpha \mathbb{Z} = \langle X,\ t \mid R,\ t^{-1}xt = x\alpha \ \ \forall x \in X \rangle;$$

also called a H-by-$\mathbb{Z}$ group. The above is called a standard presentation for H_α.

Observation

(i) *$H \trianglelefteq H_\alpha$ and $H_\alpha / H \simeq \mathbb{Z}$.*

(ii) *If $H \trianglelefteq G$ with $G/H \simeq \mathbb{Z}$, then the short exact sequence*

$$1 \longrightarrow H \longrightarrow G \longrightarrow \mathbb{Z} \longrightarrow 1$$

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Recognition of $\mathbb{Z}$-extensions

$$*\text{-by-}\mathbb{Z} = \{H \rtimes_\alpha \mathbb{Z} \mid \alpha \in \mathrm{Aut}(H)\} \subseteq \mathcal{G}$$

Given $G \in \mathcal{G}$, we have

$$\begin{aligned} G \in *\text{-by-}\mathbb{Z} \quad &\Leftrightarrow \quad \exists\, H \trianglelefteq G \text{ with } G/H \simeq \mathbb{Z} \\ &\Leftrightarrow \quad \exists\, G \twoheadrightarrow \mathbb{Z} \\ &\Leftrightarrow \quad b_1(G) \geqslant 1. \end{aligned}$$

G f.g. $\Rightarrow G^{\mathrm{ab}} = \mathbb{Z}^n \oplus T$; the first Betti number is $b_1(G) = n$.

Observation

There is an algorithm which, given $G \in \mathcal{G}$, decides whether G is a $\mathbb{Z}$-extension (of some normal subgroup $H \trianglelefteq G$).

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A group G is called a unique $\mathbb{Z}$-extension if it has a unique normal subgroup $H \trianglelefteq G$ with $G/H \simeq \mathbb{Z}$. Denote by $!$-by-$\mathbb{Z}$ the family of these groups.

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Unique extensions

Proposition (Cavallo–Kharobaei–Delgado–V.)

Let H be f.g., $b_1(H) = n$, let $\alpha \in Aut(H)$. TFAE:

(a) $H \rtimes_\alpha \mathbb{Z}$ is !-by-$\mathbb{Z}$;

(b) $b_1(H \rtimes_\alpha \mathbb{Z}) = 1$;

*(c) $\alpha^{\mathrm{ab}*} : \mathbb{Z}^n \to \mathbb{Z}^n$ has no non-trivial fixed points (say α is **deranged**)*

(d) H is a fully characteristic subgroup in $H \rtimes_\alpha \mathbb{Z}$.

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(i) H f.g. (f.p.) $\Rightarrow H \rtimes_\alpha \mathbb{Z}$ is f.g. (f.p.);
(ii) $H \rtimes_\alpha \mathbb{Z}$ f.p. $\not\Rightarrow H$ is f.g.

Proof.
(i) $H = \langle X \mid R \rangle \Rightarrow H \rtimes_\alpha \mathbb{Z} = \langle X, \; t \mid R, \; t^{-1}xt = x\alpha \;\; \forall x \in X \rangle$.

(ii) Consider a group K, take $H = *_{i\in\mathbb{Z}} K_i$ where $K_i \simeq K$, and let $\alpha \colon H \to H$, $(k \in K_i) \mapsto (k \in K_{i+1})$. We have,

$$\begin{aligned} H \rtimes_\alpha \mathbb{Z} &\simeq \langle X_i \, (i \in \mathbb{Z}), \;\; t \mid R_i, \; t^{-1}x_i t = x_{i+1} \, (i \in \mathbb{Z}, x \in X) \rangle \\ &\simeq \langle X_i \, (i \in \mathbb{Z}), \;\; t \mid R_0, \; t^{-1}x_i t = x_{i+1} \, (i \in \mathbb{Z}, x \in X) \rangle \\ &\simeq \langle X_0, \;\; t \mid R_0 \rangle \simeq K * \mathbb{Z}. \quad \square \end{aligned}$$

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Taking $1 \neq K$, f.p. and perfect ($K^{\mathrm{ab}} = 1$), we have $H = *_{i\in\mathbb{Z}} K_i$ not f.g. and so $K * \mathbb{Z}$ is f.p. and !-by-$\mathbb{Z}$, but not [f.g.]-by-$\mathbb{Z}$.

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Unrecognizability of [f.g.]-by-$\mathbb{Z}$

Theorem (Cavallo–Kharobaei–Delgado–V.)

There exists no algorithm to decide, given a finite presentation $G \in \mathcal{G}$ (even with $b_1(G) = 1$), whether $G \in$ [f.g.]-by-$\mathbb{Z}$ or not.

Proof. *There exists a recurrent sequence of finite presentations $K_j = \langle X_j \mid R_j \rangle$ such that K_j is perfect and triviality of K_j is undecidable.*

Given $j \in \mathbb{N}$,

- *$K_j * \mathbb{Z} = (*_{i \in \mathbb{Z}} K_j) \rtimes_\alpha \mathbb{Z}$ has Betti number 1;*
- *the only normal subgroup of $K_j * \mathbb{Z}$ with quotient $\mathbb{Z}$ is $\simeq *_{i \in \mathbb{Z}} K_j$;*
- *so, $K_j * \mathbb{Z} \in$ [f.g.]-by-$\mathbb{Z}$ $\Leftrightarrow$ $*_{i \in \mathbb{Z}} K_j$ f.g. $\Leftrightarrow$ $K_j = 1$, which is undecidable.* □

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Theorem (Cavallo–Kharobaei–Delgado–V.)

There exists no algorithm to decide, given a finite presentation $G \in \mathcal{G}$ (even with $b_1(G) = 1$), whether $G \in$ [f.g.]-by-$\mathbb{Z}$ or not.

Proof. *There exists a recurrent sequence of finite presentations $K_j = \langle X_j \mid R_j \rangle$ such that K_j is perfect and triviality of K_j is undecidable.*

Given $j \in \mathbb{N}$,

- $K_j * \mathbb{Z} = (*_{i \in \mathbb{Z}} K_j) \rtimes_\alpha \mathbb{Z}$ *has Betti number 1;*
- *the only normal subgroup of $K_j * \mathbb{Z}$ with quotient $\mathbb{Z}$ is $\simeq *_{i \in \mathbb{Z}} K_j$;*
- *so, $K_j * \mathbb{Z} \in$ [f.g.]-by-$\mathbb{Z}$ $\Leftrightarrow$ $*_{i \in \mathbb{Z}} K_j$ f.g. $\Leftrightarrow$ $K_j = 1$, which is undecidable.* □

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Finding standard presentations

Proposition (Cavallo–Kharobaei–Delgado–V.)

All the (finite) standard presentations of a given [f.p.]-by-$\mathbb{Z}$ group G are recursively enumerable.

Proof. *We are given a finite presentation $\langle X \mid R \rangle$ of a group G which is in [f.p.]-by-$\mathbb{Z}$.*

• *Enumerate all pres. of G (by diagonally applying all possible Tietze transformations to $\langle X \mid R \rangle$), of the form*

$$\langle y_1, \ldots, y_n, t \mid r_i,\ \ t^{-1} y_j t = w_j\ (i = 1, \ldots, m), (j = 1, \ldots, n) \rangle,$$

where the r_i's and w_j's are words on the y_j's.

• *For each such pres., check whether $y_j \mapsto w_j$ defines an endo, say α, of $H = \langle y_1, \ldots, y_n \mid r_1, \ldots, r_m \rangle$ (by enumerating $\ll r_1, \ldots, r_m \gg$ and checking whether each $r_i(w_1, \ldots, w_n)$ does appear in the list).*
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- *Each such process will either output a valid standard presentation for G, or work forever with no output.*
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Outline

The BNS invariant

The theory of sigma invariants was started and developed in the 1980's by Robert Bieri, Walter Neumann and Ralf Strebel

Definition

Let $G = \langle X \mid R \rangle$ be a f.g. group. A character is a morphism $\chi\colon G \to \mathbb{R}$. Every such χ factors through $G^{\mathrm{ab}*} = G^{\mathrm{ab}}/T(G^{\mathrm{ab}}) = \mathbb{Z}^n$ (where $n = b_1(G)$) and so,

$$\{\text{characters of } G\} = \mathrm{Hom}(G, \mathbb{R}) = \mathrm{Hom}(\mathbb{Z}^n, \mathbb{R}) \simeq \mathbb{R}^n.$$

Define $\chi_1 \sim \chi_2 \;\Leftrightarrow\; \chi_2 = \lambda\chi_1$ *for some* $\lambda > 0$*,*

$$S(G) = \{\chi\colon G \to \mathbb{R} \mid \chi \neq 0\}/\sim \;= (\mathbb{R}^n \setminus \{0\})/\sim \;= S^{n-1}.$$

is the character sphere of G. Given $H \leqslant G$ define

$$S(G, H) = \{\chi \in S(G) \mid \chi(H) = 0\}.$$

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The first sigma invariant of G (also called the BNS invariant) is the following subset of the character sphere:

$$\Sigma^1(G) = \{\chi \in S(G) \mid G_\chi \text{ is connected in } \Gamma(G, X)\} \subseteq S(G),$$

where $G_\chi = \{g \in G \mid \chi(g) > 0\}$ is the positive cone; (this connectivity does not depend on X !).

Theorem

Let G be f.g. and $H \trianglelefteq G$ s.t. G/H is abelian. Then,

$$H \text{ is f.g.} \quad \Leftrightarrow \quad S(G, H) \subseteq \Sigma^1(G).$$

In particular, if $G/H = \mathbb{Z}$ and $\pi\colon G \twoheadrightarrow G/H = \mathbb{Z}$, then

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Undecidability of the BNS invariant

Theorem (Cavallo–Kharobaei–Delgado–V.)

There exists no algorithm s.t., given a finite pres. $G = \langle X \mid R \rangle$, and a character $\chi\colon G \to \mathbb{R}$ (i.e., a point $p = [\chi] \in S(G)$) decides whether $p \in \Sigma^1(G)$ or not.

Proof. *Suppose there exists such an algorithm $\mathcal{A}$.*

Consider any finite pres. $G = \langle X \mid R \rangle \in$!-by-$\mathbb{Z}$ (i.e., with $b_1(G) = 1$), and let $\pi\colon G \twoheadrightarrow G^{\mathrm{ab}*} = \mathbb{Z}$.

Apply $\mathcal{A}$ to G and both $\pm\pi$ to decide whether $\pi \in \Sigma^1(G)$ or not, and whether $-\pi \in \Sigma^1(G)$ or not.

But, $\pm\pi \in \Sigma^1(G) \;\Leftrightarrow\; \ker(\pi)$ is f.g. $\;\Leftrightarrow\; G \in$ [f.g.]-by-$\mathbb{Z}$.

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Question

Given $H = \langle X \mid R \rangle$ *and* $\alpha, \beta \in Aut(H)$: $\quad H \rtimes_\alpha \mathbb{Z} \simeq H \rtimes_\beta \mathbb{Z} \quad \Leftrightarrow \quad ?$

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$$[\alpha] \sim [\beta]^{\pm 1} \text{ in } Out(H) \quad \Rightarrow \quad H \rtimes_\alpha \mathbb{Z} \simeq H \rtimes_\beta \mathbb{Z}.$$

($\alpha = \chi^{-1}\beta^{\pm 1}\chi\gamma_h$ *for some* $h \in H$ *and* $\chi \in Aut(H)$)

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For $H = F_2$ *and* $\alpha, \beta \in Aut(F_2)$*:*
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Example (Dicks)

$\exists\, \alpha, \beta \in Aut(F_3)$ *such that* $F_3 \rtimes_\alpha \mathbb{Z} \simeq F_3 \rtimes_\beta \mathbb{Z}$ *but* $[\alpha] \not\sim [\beta]^{\pm 1}$ *in* $Out(F_3)$.

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A solution for the deranged case

Observation

Let H and K be f.g. and $\alpha \in Aut(H)$, $\beta \in Aut(K)$ *be deranged. Then,*

(i) $H \rtimes_\alpha \mathbb{Z} \simeq K \rtimes_\beta \mathbb{Z} \quad \Rightarrow \quad H \simeq K$;

(ii) all isomorphisms from $H \rtimes_\alpha \mathbb{Z}$ to $H \rtimes_\beta \mathbb{Z}$ (if any) are of the form:

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where $\psi \in Aut(H)$, $\epsilon = \pm 1$ and $h \in H$ such that $\psi\beta^\epsilon\gamma_h = \alpha\psi$;

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A solution for the deranged case

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The isomorphism problem

Theorem (Cavallo–Kharobaei–Delgado–V.)

Let $\mathcal{H} \subseteq \mathcal{G}$ be a family of f.p. groups. Then,

$$\forall H \in \mathcal{H}, \quad \left. \begin{array}{c} \mathrm{Isom}(\mathcal{H}) \textit{ solvable} \\ \frac{1}{2}\mathrm{CP}'(Out(H)) \textit{ solvable} \end{array} \right| \Rightarrow \quad \mathrm{Isom}(!\mathcal{H}\text{-by-}\mathbb{Z}) \textit{ solvable.}$$

Definition

For a group G, the $\frac{1}{2}\mathrm{CP}(G)$ consists on deciding, given $g_1, g_2 \in G$, whether $g_1 \sim g_2^{\pm 1}$.

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$\mathrm{CP}(G)$ *solvable* $\Rightarrow$ $\frac{1}{2}\mathrm{CP}(G)$ *solvable.*

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For a f.p. group $H = \langle X \mid R \rangle$, suppose we know a finite set of autos $\alpha_1, \ldots, \alpha_n \in Aut(H)$ generating $Out(H)$. Then,

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Let $\mathcal{F}$ be the family of f.g. free groups. Modulo a solution to $\mathrm{CP}(Out(F_n))$ *for all* $n \in \mathbb{N}$, $\mathrm{Isom}(!\mathcal{F}\text{-by-}\mathbb{Z})$ *is solvable.*

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Let $\mathcal{B}$ be the family of Braid groups, $\mathcal{B} = \{B_n \mid n \geqslant 2\}$. Then, $\mathrm{Isom}(!\mathcal{B}\text{-by-}\mathbb{Z})$ *is solvable.*

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