Composition operators induced by universal covering maps

Joint International Meeting AMS, EMS, and SPM

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13th June 2015
Preliminaries

- $\mathbb{D} = \{z : |z| < 1\}$
- $H^p$ $(1 \leq p < \infty)$ the Hardy space, $f \in H^p$ if and only if
  \[
  \|f\|_p^p = \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta < \infty
  \]
- If $\phi: \mathbb{D} \to \mathbb{D}$ is holomorphic then the composition operator $C_\phi$ is
  \[
  C_\phi : H^p \to H^p
  \]
  \[
  f \mapsto f \circ \phi
  \]
Boundedness of $C_\phi$

Theorem (Littlewood’s subordination theorem (1925))

Suppose that $\phi: \mathbb{D} \rightarrow \mathbb{D}$ is univalent and holomorphic, $\phi(0) = 0$. If $f \in H^p$ ($1 \leq p < \infty$) then

$$\int_0^{2\pi} |f(\phi(re^{i\theta}))|^p d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^p d\theta$$

So $C_\phi: H^p \rightarrow H^p$ is a bounded operator.
Theorem (Littlewood’s subordination theorem (1925))

Suppose that \( \phi: \mathbb{D} \to \mathbb{D} \) is univalent and holomorphic, \( \phi(0) = 0 \). If \( f \in H^p \) \((1 \leq p < \infty)\), then

\[
\int_0^{2\pi} |f(\phi(re^{i\theta}))|^p d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^p d\theta
\]

So

\[
C_\phi : H^p \to H^p
\]

is a bounded operator.
Compactness of $C_\phi$

If $\phi : \mathbb{D} \to \mathcal{D}_0$ is univalent then $C_\phi$ is compact if and only if

$$\lim_{|z| \to 1} \frac{1 - |\phi(z)|}{1 - |z|} = \infty.$$
If $\phi: \mathbb{D} \rightarrow D_0$ is univalent then $C_\phi$ is compact if and only if

$$\lim_{|z| \to 1} \frac{1 - |\phi(z)|}{1 - |z|} = \infty.$$ 

The angular derivative does not exist anywhere.
Consider instead domains of the form

\[ D = D_0 \setminus \{ p_1, p_2, \ldots, p_n \} \]

where \( p_k, k = 1, 2, \ldots n \) are isolated points in \( D_0 \).
The uniformization theorem

- For simply connected $\mathcal{D}_0$ we have the Riemann mapping theorem.
- There is a ‘unique’ univalent mapping $\psi$ of $\mathbb{D}$ onto $\mathcal{D}_0$.

- For domains $\mathcal{D} = \mathcal{D}_0 \setminus \{p_1, p_2, \ldots, p_n\}$ we must employ the uniformization theorem.
- There is a holomorphic universal covering map of $\mathbb{D}$ onto $\mathcal{D}$.
The universal covering map

\[ \mathbb{D} \xrightarrow{\phi} D \xrightarrow{\pi} \mathcal{R}_D \cong \mathbb{D}/\Gamma \]

\[ \mathbb{D} \xrightarrow{\tilde{\phi}_D} \mathcal{R}_D \]

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Riemann mapping from $\mathbb{D}$ to $\mathbb{D}$ is $\psi(z) = z$

A universal covering map from $\mathbb{D}$ to $\mathbb{D}\setminus\{0\}$ is

$$\phi(z) = \exp\left(-\frac{1+z}{1-z}\right)$$

Note $\phi$ is an inner function with

$$\lim_{r \to 1^-} |\phi(re^{i\theta})| = \begin{cases} 0, & \theta = 0 \\ 1, & \text{otherwise} \end{cases}$$
Definition (Nevanlinna’s counting function)

For any function $\phi: \mathbb{D} \to \mathbb{D}$

$$\mathcal{N}_\phi(w) = \begin{cases} \sum_{z: \phi(z) = w} \log \frac{1}{|z|} & w \in \phi(\mathbb{D}) \\ 0 & w \in \mathbb{D} \setminus \phi(\mathbb{D}) \end{cases}$$
Definition (Nevanlinna’s counting function)

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$$
\mathcal{N}_\phi(w) = \begin{cases} 
\sum_{z : \phi(z)=w} \log \frac{1}{|z|} & w \in \phi(\mathbb{D}) \\
0 & w \in \mathbb{D} \setminus \overline{\phi(\mathbb{D})}
\end{cases}
$$

Theorem (Shapiro’s compactness criterion)

$C_\phi$ is compact on $H^p$ if and only if

$$
\lim_{|w| \to 1} \frac{\mathcal{N}_\phi(w)}{\log 1/|w|} = 0
$$
Proof of results

For our universal covering map

\[ \phi: \mathbb{D} \to \mathbb{D} = \mathbb{D}_0 \setminus \{p_1, \ldots, p_n\} \]

we have:

- \( \mathcal{R}_\mathbb{D} \simeq \mathbb{D}/\Gamma \) for \( \Gamma \) a torsion-free Fuchsian group
- \( \Lambda(\Gamma) \) – the limit set of \( \Gamma \) – satisfies \( \Lambda(\Gamma) \not\subseteq \partial \mathbb{D} \)
- \( \phi^{-1}(w) \) is a \( \Gamma \)-orbit
- \( z_1, z_2 \in \phi^{-1}(w) \) if and only if \( \exists h \in \Gamma \) such that \( z_1 = h(z_2) \).
Main results

Theorem

Suppose that $\mathcal{D} = \mathcal{D}_0 \setminus \{p_1, \ldots, p_n\}$ and $\phi$ is a universal covering of $\mathbb{D}$ onto $\mathcal{D}$. Then $C_\phi$ is compact on $H^p$, $1 \leq p < \infty$, if and only if

$$\lim_{z \to \zeta} \frac{1 - |\phi(z)|}{1 - |z|} = \infty$$

for all $\zeta \in \partial \mathbb{D}$. 
Main results

Theorem

Suppose that $\mathcal{D} = \mathcal{D}_0 \backslash \{p_1, \ldots, p_n\}$ and $\phi$ is a universal covering of $\mathbb{D}$ onto $\mathcal{D}$. Then $C_\phi$ is compact on $H^p$, $1 \leq p < \infty$, if and only if

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for all $\zeta \in \partial \mathbb{D}$.

Theorem

Suppose that $\mathcal{D} = \mathcal{D}_0 \backslash \{p_1, \ldots, p_n\}$, $\phi$ is a universal covering of $\mathbb{D}$ onto $\mathcal{D}$, and $\psi$ is the univalent Riemann mapping of $\mathbb{D}$ onto $\mathcal{D}_0$. Then $C_\phi$ is compact on $H^p$, $1 \leq p < \infty$, if and only if $C_\psi$ is.
Definition (The Poincare series)

For a Fuchsian group $\Gamma$ and $z, w \in \mathbb{D}$

$$\rho_{\Gamma}(z, w; s) = \sum_{g \in \Gamma} \exp(-sd_{\mathbb{D}}(z, g(w)))$$

where

$$d_{\mathbb{D}}(z, w)$$

is the hyperbolic distance in $\mathbb{D}$. 
Proof of results

Lemma

There are constants $c_1$ and $c_2$ such that for $w = \phi(z)$ suitably chosen

$$c_1 \rho \Gamma(0, z; 1) \leq \mathcal{N}_\phi(w) \leq c_2 \rho \Gamma(0, z; 1)$$
Proof of result

Lemma

\( C_\phi \) is compact on \( H^p \) if and only if for all \( \zeta \not\in \Lambda(\Gamma) \)

\[
\lim_{z \to \zeta} \frac{\rho_\Gamma(0, z; 1)}{1 - |\phi(z)|} = 0
\]

Lemma

If \( \Gamma \) is finitely generated then there are constants \( c_1 \) and \( c_2 \) such that for \( z \) close enough to \( \partial \mathbb{D} \setminus \Lambda(\Gamma) \)

\[
c_1(1 - |z|^2) \leq \rho_\Gamma(0, z; 1) \leq c_2(1 - |z|^2)
\]
Proof of main result

1. \( C_\phi \) compact if and only if for all \( \zeta \notin \Lambda(\Gamma) \)

\[
\lim_{z \to \zeta} \frac{\rho_\Gamma(0, z; 1)}{1 - |\phi(z)|} = \lim_{z \to \zeta} \frac{1 - |z|}{1 - |\phi(z)|} = 0
\]

2. \( \Lambda(\Gamma) \) consists of:
   - fixed points of parabolic elements (correspond to \( p_i \))
   - points of approximation (where orbits converge non-tangentially and therefore \( \phi \) is fixed on a sequence converging non-tangentially)
These results can be extended to general (finitely) multiply connected domains.

Widely applicable in the study of composition operators since the Nevanlinna counting function appears often.

The characterisation of Schatten class composition operators in particular employs $\mathcal{N}_\phi$. 