Linear spaces of matrices of constant rank and vector bundles

Emilia Mezzetti

Dipartimento di Matematica e Geoscienze
Università degli studi di Trieste
mezzette@units.it

Vector bundles on Projective Varieties / Oporto, 10/06/2015

Outline

Notations

K algebraically closed field, $\mathrm{char}(K) = 0$

A $m \times n$ matrix of linear forms in d variables over K

Interpretation:
Fix V, W, $\dim(V) = n$, $\dim(W) = m$
K-vector spaces and bases

$A \subset Hom(V^*, W) \simeq V \otimes W$: vector subspace of dimension d

Hypothesis:

- $d \geq 2$
- A of constant rank r, i.e.
  every non-zero matrix obtained specializing the variables has rank r

Notations

K algebraically closed field, $\mathrm{char}(K) = 0$

A $m \times n$ matrix of linear forms in d variables over K

Interpretation:
Fix V, W, $\dim(V) = n$, $\dim(W) = m$
K-vector spaces and bases

$A \subset Hom(V^*, W) \simeq V \otimes W$: vector subspace of dimension d

Hypothesis:

- $d \geq 2$
- A of constant rank r, i.e.
  every non-zero matrix obtained specializing the variables has rank r

Problems

Definition

$I(r, m, n) = \max\{\dim(A) \mid A\ m \times n \text{ matrix of constant rank } r\}$

Problems

1. Compute $I(r, m, n)$
2. For $d \leq I(r, m, n)$ "classify" the linear systems A under the action of $GL(V) \times GL(W)$ by strict equivalence: $A' \sim A$ if $A' = MAN$ (or other actions).

Answer known only in particular cases, for instance

- $d = 2$ pencils of matrices
- $r = 1$ $\mathbb{P}(A) \subset X := \mathbb{P}(V) \times \mathbb{P}(W)$

Problems

Definition

$l(r, m, n) = \max\{\dim(A) \mid A\ m \times n \text{ matrix of constant rank } r\}$

Problems

1. Compute $l(r, m, n)$
2. For $d \leq l(r, m, n)$ "classify" the linear systems A under the action of $GL(V) \times GL(W)$ by strict equivalence:
   $A' \sim A$ if $A' = MAN$ (or other actions).

Answer known only in particular cases, for instance

- $d = 2$ pencils of matrices
- $r = 1$ $\mathbb{P}(A) \subset X := \mathbb{P}(V) \times \mathbb{P}(W)$

Problems

Definition

$l(r,m,n) = \max\{\dim(A) \mid A \ m \times n \ \text{matrix of constant rank } r\}$

Problems

1. Compute $l(r,m,n)$
2. For $d \leq l(r,m,n)$ "classify" the linear systems A under the action of $GL(V) \times GL(W)$ by strict equivalence: $A' \sim A$ if $A' = MAN$ (or other actions).

Answer known only in particular cases, for instance

- $d = 2$ pencils of matrices
- $r = 1$ $\mathbb{P}(A) \subset X := \mathbb{P}(V) \times \mathbb{P}(W)$

Secant varieties

$$X \subset SX \subset S^2X \subset \cdots \subset \mathbb{P}(V \otimes W)$$

stratification by rank

Known facts

$\sim$ 1980, J. Sylvester, R. Westwick,...

If $2 \leq r \leq m \leq n$

$n - r + 1 \leq l(r, m, n) \leq n + m - 2r + 1$

Assumption

$n = m$, $V = W$ and A symmetric or skew-symmetric

1999, B. Ilic - J. Landsberg

$\max\{\dim(A) \mid A\ n \times n \text{ symmetric of constant rank } r\} = n - r + 1$

No classification

Known facts

$\sim$ 1980, J. Sylvester, R. Westwick,...

If $2 \leq r \leq m \leq n$

$n - r + 1 \leq l(r, m, n) \leq n + m - 2r + 1$

Assumption

$n = m$, $V = W$ and A symmetric or skew-symmetric

1999, B. Ilic - J. Landsberg

$\max\{\dim(A) \mid A\ n \times n \text{ symmetric of constant rank } r\} = n - r + 1$

No classification

Known facts

$\sim$ 1980, J. Sylvester, R. Westwick,...

If $2 \leq r \leq m \leq n$

$n - r + 1 \leq l(r, m, n) \leq n + m - 2r + 1$

Assumption
$n = m$, $V = W$ and A symmetric or skew-symmetric

1999, B. Ilic - J. Landsberg

$\max\{\dim(A) \mid A \; n \times n \text{ symmetric of constant rank } r\} = n - r + 1$

No classification

Outline

1. Introduction and problems
2. Skew-symmetric matrices
   - Associated bundles
   - $\dim(A) = 3$
   - $\dim(A) > 3$

Exact sequence

Consider A skew-symmetric

$X = \mathbb{G}(1, \mathbb{P}(V)) \subset \mathbb{P}(\Lambda^2 V)$

action of $GL(V)$ by conjugation

A defines an exact sequence

$$0 \to K \to V^* \otimes \mathcal{O}_{\mathbb{P}(A)} \to V \otimes \mathcal{O}_{\mathbb{P}(A)}(1) \to N \to 0$$

- A has constant rank r if and only if K and N are vector bundles of rank $n - r$
- A symmetric or skew-symmetric: $N \simeq K^*(1)$
- $c_1(K^*) = \frac{r}{2}$.

Exact sequence

Consider A skew-symmetric

$X = \mathbb{G}(1, \mathbb{P}(V)) \subset \mathbb{P}(\Lambda^2 V)$

action of $GL(V)$ by conjugation

A defines an exact sequence

$$0 \to K \to V^* \otimes \mathcal{O}_{\mathbb{P}(A)} \to V \otimes \mathcal{O}_{\mathbb{P}(A)}(1) \to N \to 0$$

- A has constant rank r if and only if K and N are vector bundles of rank $n - r$
- A symmetric or skew-symmetric: $N \simeq K^*(1)$
- $c_1(K^*) = \frac{r}{2}$.

Example

$r = 2$
$A \subset \mathbb{G}(1, \mathbb{P}(V))$ maximal

- A represents the lines through a point, $K^* = T_{\mathbb{P}A}(-1) = Q$

$$\begin{pmatrix} 0 & a_1 & a_2 & \cdots & a_d \\ -a_1 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_d & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

- A represents the lines of a 2-plane, $K^* = \mathcal{O}_{\mathbb{P}A}(1)$

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}.$$

Corank 2

$n - r = 0, 1$: easy

Assumption: $n - r = 2$

- rank $K = 2$
- K^* generated by global sections
- K^* defines an embedding $\mathbb{P}(A) \to \mathbb{G}(1, \mathbb{P}(V))$

For any even r there exists a linear system A of skew-symmetric matrices of rank r and corank 2 with dim$(A) = 3$.

Corank 2

$n - r = 0, 1$: easy

Assumption: $n - r = 2$

- rank $K = 2$
- K^* generated by global sections
- K^* defines an embedding $\mathbb{P}(A) \to \mathbb{G}(1, \mathbb{P}(V))$

For any even r there exists a linear system A of skew-symmetric matrices of rank r and corank 2 with $\dim(A) = 3$.

Corank 2

$n - r = 0, 1$: easy

Assumption: $n - r = 2$

- rank $K = 2$
- K^* generated by global sections
- K^* defines an embedding $\mathbb{P}(A) \to \mathbb{G}(1, \mathbb{P}(V))$

For any even r there exists a linear system A of skew-symmetric matrices of rank r and corank 2 with $\dim(A) = 3$.

Outline

Skew-symmetric matrices of corank 2 with $d = 3$

A rank 2 vector bundle on $\mathbb{P}^2$ is called *m*-**effective** if it is of the form K^* for some linear system A of skew-symmetric matrices of constant rank.

- $r = 4$: there are four orbits of 2-planes of 6×6 matrices of constant rank 4.
  All globally generated rk 2 bundles on $\mathbb{P}^2$ with $c_1 = 2$, define an embedding in $\mathbb{G}(1,5)$ and are *m*-effective [Manivel - M, 2005]:
  $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)$, $\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$, Steiner bundle, null-correlation bundle restricted.
- $r = 6$: every gg rk 2 bundle with $c_1 = 3$, defining an embedding of $\mathbb{P}^2$ in $\mathbb{G}(1,7)$, is *m*-effective [Fania - M, 2011] (8×8 matrices of constant rank 6).

Question

Assume $d = 3$.

Question

Determine all the pairs (c_1, c_2) such that there exists a globally generated vector bundle E of rank 2 on $\mathbb{P}^2$, with $c_1(E) = c_1$ and $c_2(E) = c_2$, such that $E = K^*$ for a linear system A of skew-symmetric matrices of constant rank $r = 2c_1$ and size $n = r + 2$.

[Boralevi - M, 2015]

Question

Assume $d = 3$.

Question

Determine all the pairs (c_1, c_2) such that there exists a globally generated vector bundle E of rank 2 on $\mathbb{P}^2$, with $c_1(E) = c_1$ and $c_2(E) = c_2$, such that $E = K^*$ for a linear system A of skew-symmetric matrices of constant rank $r = 2c_1$ and size $n = r + 2$.

[Boralevi - M, 2015]

m-effective rank 2 bundles on $\mathbb{P}^2$

The answer is based on a result of Ph.Ellia (2013): description of all **effective** pairs (c_1, c_2), such that there exists E globally generated of rank 2 on $\mathbb{P}^2$, with $c_1(E) = c_1$ and $c_2(E) = c_2$.

Assume $c_1 > 0$, $c_2 > 0$.
Consider separately **stable range** and **unstable range**.

m-effective rank 2 bundles on $\mathbb{P}^2$

The answer is based on a result of Ph.Ellia (2013): description of all **effective** pairs (c_1, c_2), such that there exists E globally generated of rank 2 on $\mathbb{P}^2$, with $c_1(E) = c_1$ and $c_2(E) = c_2$.

Assume $c_1 > 0$, $c_2 > 0$.
Consider separately **stable range** and **unstable range**.

Stable range

Assume $c_1^2 - 4c_2 < 0$, and moreover the necessary condition $c_2 \leq c_1^2$.

- Every pair is effective (Le Potier)
- (c_1, c_2) is m-effective if and only if $c_2 \leq \binom{c_1+1}{2}$
- If equality holds, E is a Steiner bundle.

Stable range

Assume $c_1^2 - 4c_2 < 0$, and moreover the necessary condition $c_2 \le c_1^2$.

- Every pair is effective (Le Potier)
- (c_1, c_2) is *m*-effective if and only if $c_2 \le \binom{c_1+1}{2}$
- If equality holds, E is a Steiner bundle.

Stable range

Assume $c_1^2 - 4c_2 < 0$, and moreover the necessary condition $c_2 \leq c_1^2$.

- Every pair is effective (Le Potier)
- (c_1, c_2) is m-effective if and only if $c_2 \leq \binom{c_1+1}{2}$
- If equality holds, E is a Steiner bundle.

Unstable range

Assume $c_1^2 - 4c_2 > 0$.

- There are gaps, not all pairs (c_1, c_2) are effective.
- There exist effective pairs (c_1, c_2) that are not associated to any *m*-effective bundle.
- There exist effective bundles E even defining an embedding in $\mathbb{G}(1, n-1)$ but not *m*-effective.
- First class of examples: $(c_1, 2c_1)$ with $c_1 \geq 10$.

Unstable range

Assume $c_1^2 - 4c_2 > 0$.

- There are gaps, not all pairs (c_1, c_2) are effective.
- There exist effective pairs (c_1, c_2) that are not associated to any m-effective bundle.
- There exist effective bundles E even defining an embedding in $\mathbb{G}(1, n-1)$ but not m-effective.
- First class of examples: $(c_1, 2c_1)$ with $c_1 \geq 10$.

Unstable range

Assume $c_1^2 - 4c_2 > 0$.

- There are gaps, not all pairs (c_1, c_2) are effective.
- There exist effective pairs (c_1, c_2) that are not associated to any m-effective bundle.
- There exist effective bundles E even defining an embedding in $\mathbb{G}(1, n-1)$ but not m-effective.
- First class of examples: $(c_1, 2c_1)$ with $c_1 \geq 10$.

Unstable range

Assume $c_1^2 - 4c_2 > 0$.

- There are gaps, not all pairs (c_1, c_2) are effective.
- There exist effective pairs (c_1, c_2) that are not associated to any m-effective bundle.
- There exist effective bundles E even defining an embedding in $\mathbb{G}(1, n-1)$ but not m-effective.
- First class of examples: $(c_1, 2c_1)$ with $c_1 \geq 10$.

Quotient of direct sums

All known examples of *m*-effective bundles *E* are "quotients":

$$0 \to \mathcal{O}_{\mathbb{P}^2}^k \to F \to E \to 0$$

with

$$F = (\oplus_{i \geq 0} \mathcal{O}_{\mathbb{P}^2}(i)^{a_i}) \oplus T_{\mathbb{P}^2}(-1)^b$$

$a_i, b \geq 0$.

Every direct summand gives a building block: we construct a matrix of rank *r* direct sum of building blocks, then perform a suitable projection to get corank 2.

Quotient of direct sums

All known examples of *m*-effective bundles *E* are “quotients”:

$$0 \to \mathcal{O}_{\mathbb{P}^2}^k \to F \to E \to 0$$

with

$$F = (\oplus_{i \geq 0} \mathcal{O}_{\mathbb{P}^2}(i)^{a_i}) \oplus T_{\mathbb{P}^2}(-1)^b$$

$a_i, b \geq 0$.

Every direct summand gives a building block: we construct a matrix of rank r direct sum of building blocks, then perform a suitable projection to get corank 2.

Outline

$\dim(A) > 3$

Do there exist linear systems A of skew-symmetric matrices of constant corank 2 with $\dim(A) > 3$?

Westwick's example (1996): 10×10 skew-symmetric matrix of constant rank 8 and dimension 4.

[A. Boralevi, D. Faenzi, M, 2013]

- r must be of the form $12s$ or $12s - 4$;
- suppose there exists A having K as kernel: write $K = E(-\frac{r}{4} - 2)$, then E is a vector bundle on $\mathbb{P}^3$ with $c_1(E) = 0$, $c_2(E) = \frac{r(r+4)}{48}$.

$\dim(A) > 3$

Do there exist linear systems A of skew-symmetric matrices of constant corank 2 with $\dim(A) > 3$?

Westwick's example (1996): 10×10 skew-symmetric matrix of constant rank 8 and dimension 4.

[A. Boralevi, D. Faenzi, M, 2013]

- r must be of the form $12s$ or $12s - 4$;
- suppose there exists A having K as kernel: write $K = E(-\frac{r}{4} - 2)$, then E is a vector bundle on $\mathbb{P}^3$ with $c_1(E) = 0$, $c_2(E) = \frac{r(r+4)}{48}$.

$\dim(A) > 3$

Do there exist linear systems A of skew-symmetric matrices of constant corank 2 with $\dim(A) > 3$?

Westwick's example (1996): 10×10 skew-symmetric matrix of constant rank 8 and dimension 4.

[A. Boralevi, D. Faenzi, M, 2013]

- r must be of the form $12s$ or $12s - 4$;
- suppose there exists A having K as kernel: write $K = E(-\frac{r}{4} - 2)$, then E is a vector bundle on $\mathbb{P}^3$ with $c_1(E) = 0$, $c_2(E) = \frac{r(r+4)}{48}$.

Instantons

The exact sequence has the fom

$$0 \to E(-\frac{r}{4} - 2) \to \mathcal{O}_{\mathbb{P}^3}(-2)^{r+2} \to \mathcal{O}_{\mathbb{P}^3}(-1)^{r+2} \to E(\frac{r}{4} - 1) \to 0.$$

This is a 2-extension, so it gives a class $\beta \in Ext^2(E(\frac{r}{4} - 1), E(-\frac{r}{4} - 2))$.

- We determine necessary and sufficient cohomological conditions on a a bundle E and 2-extension β , to produce a 2-term complex of the desired form with A skew-symmetric.

Instantons

The exact sequence has the fom

$$0 \to E(-\frac{r}{4} - 2) \to \mathcal{O}_{\mathbb{P}^3}(-2)^{r+2} \to \mathcal{O}_{\mathbb{P}^3}(-1)^{r+2} \to E(\frac{r}{4} - 1) \to 0.$$

This is a 2-extension, so it gives a class $\beta \in Ext^2(E(\frac{r}{4} - 1), E(-\frac{r}{4} - 2))$.

- We determine necessary and sufficient cohomological conditions on a a bundle E and 2-extension β , to produce a 2-term complex of the desired form with A skew-symmetric.

Application

Application

There exists β verifying the conditions if E is:

- any 2-instanton, then $r = 8$, A is a 10×10 matrix;
- a general 4-instanton, then $r = 12$, A is a 14×14 matrix.

Explicit constructions:

[A. Boralevi - D. Faenzi - P. Lella (2015)]

$\dim(A) = 4$

Do there exist any $\mathbb{P}^4$ of skew-symmetric matrices of constant corank 2?

- The first possible case would have $r = 32$
- E cannot split
- E cannot be a Horrocks-Mumford bundle