Theta divisors of stable vector bundles with many maximal subbundles

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Stability of vector bundles over curves

Let X be a complex projective smooth curve of genus $g \geq 0$.

Recall that a vector bundle $W \to X$ is *semistable* if for every proper subbundle $F \subset W$, we have

$$\frac{\deg F}{\operatorname{rank} F} \leq \frac{\deg W}{\operatorname{rank} W}.$$

If inequality is strict for all F, then V is *stable*.

Generalised theta divisors

Write J_X^{g-1} for the Jacobian variety parametrising line bundles over X of degree $g-1$.

Let $W \to X$ be a vector bundle of rank r and trivial determinant.

Consider the set $\{P \in J_X^{g-1} : h^0(X, P \otimes W) > 0\} \subseteq J_X^{g-1}$.

This has expected codimension one, since $\chi(X, P \otimes W) = 0$.

It is by now well known that for general semistable W, this is the support of a divisor $\mathcal{D}(W)$ on J_X^{g-1}, linearly equivalent to $r\Theta$.

Examples

1. The trivial bundle: $\mathcal{D}(\mathcal{O}_X)$ is the Riemann theta divisor

$$W_{g-1} \;=\; \{\mathcal{O}_X(D) : D \in \mathrm{Sym}^{g-1}X\}.$$

2. (Beauville, 2006) If $W \cong W^*$ then $\mathcal{D}(W) = \iota^*\mathcal{D}(W)$ where $\iota(L) = K_X L^{-1}$. More precisely:

- If W is symplectic of rank $2n$ then $\mathcal{D}(W) \in |2n\Theta|_+$.

- If W is orthogonal of rank r, then $\mathcal{D}(W) \in |r\Theta|_\pm$, depending on the second Stiefel–Whitney class of W.

Bundles with nonintegral theta divisors

Let W_1 and W_2 be stable bundles of trivial determinant.

Consider an extension $0 \to W_1 \to W \to W_2 \to 0$.

Then $\mathcal{D}(W)$ is the *reducible* divisor $\mathcal{D}(W_1) + \mathcal{D}(W_2)$, when this exists.

If $W_2 = W_1$, then $\mathcal{D}(W) = 2 \cdot \mathcal{D}(W_1)$ is *nonreduced*.

As $\deg W_1 = 0 = \deg W$, such a W is strictly semistable.

Question: Can $\mathcal{D}(W)$ fail to be integral if W is *stable*?

First example (Raynaud, 1990s; Pauly, 2014)

Suppose $f\colon X \to Z$ is a double cover of an elliptic curve.

Then there exists a stable bundle $W \to X$ of rank two and trivial determinant, with theta divisor $\mathcal{D}(W)$ birationally equivalent to

$$\left(Z \times \mathrm{Sym}^{g-2} X\right) \ \cup \ \mathrm{Prym}(X/Z).$$

The component $Z \times \mathrm{Sym}^{g-2} X$ arises because of a family of maximal line subbundles in W of degree -1 parametrised by Z.

Further examples (Beauville, 2003)

Let X be any curve, and $L \to X$ be a general very ample line bundle of degree $2g$.

Consider the evaluation sequence,

$$0 \to E_L^* \to \mathcal{O}_X \otimes H^0(X, L) \to L \to 0.$$

The bundle E_L has slope 2, so $\mathcal{D}(E_L)$ lives in J_X^{g-3}. Beauville shows that E_L is stable and that

$$\mathcal{D}(E_L) = \left(W_{g-2} - W_1\right) \cup \mathcal{D}\left(K_X^{-1} L\right),$$

where W_d is the locus of degree d bundles with nonzero sections:

$$W_d := \{\mathcal{O}_X(D) : D \text{ an effective divisor of degree } d\}.$$

As in Raynaud's example, the component $W_{g-2} - W_1$ arises from a family of maximal subbundles $\{\mathcal{O}_X(p) : p \in X\}$ in E_L.

Again, these subbundles have maximal possible degree given that E_L is stable.

Beauville also shows that for $1 \leq p < g$, the exterior power $\bigwedge^p E_L$ is stable and satisfies

$$\mathcal{D}\left(\bigwedge^p E_L\right) = \left(W_{g-p-1} - W_p\right) + \left(W_{g-p} - W_{p-1} + K_X L^{-1}\right)$$

in J_X^{g-1-2p}.

Corollary (Beauville, 2003): The theta map $\mathcal{D}\colon \mathcal{SU}_r \dashrightarrow |r\Theta|$ is not injective for $r = g$.

Proof
The strictly semistable bundle $E_{K_X} \oplus K_X L^{-1}$ has the same theta divisor as the stable bundle E_L.

Translating by a gth root of L^{-1}, we obtain two nonisomorphic semistable bundles of trivial determinant with the same theta divisor in $|g\Theta|$. $\square$

Bundles with reducible and nonreduced theta divisors

Theorem (H., 2014): Suppose $g \geq 5$ and $n \geq 1$. Let t be such that $2 \leq t < n(g-1)$. Let $\mathcal{D}(V)$ be the theta divisor of a generic bundle V of rank n and degree zero.

Then for all $r \geq tn+2$, there exists a stable rank r bundle W such that $\mathcal{D}(W)$ is defined and contains $(t-1)\cdot\mathcal{D}(V)$ as a subscheme.

Corollary: If $g \geq 5$, then for all $r \geq 4$ (resp., $r \geq 5$), there exist stable bundles of rank r with reducible (resp., reducible and nonreduced) theta divisors.

Idea of construction

Let V be a general bundle of degree zero and rank n and L a general line bundle of degree -1. Choose a general extension

$$0 \to L \to E \to V^{\oplus t} \to 0$$

For general $P \in \mathcal{D}(V)$, we have

$$0 \to H^0(E \otimes P) \to H^0(V \otimes P)^{\oplus t} \to H^1(L \otimes P) \to \cdots$$

As $h^1(L \otimes P) = 1$, we have $h^0(E \otimes P) \geq t - 1$.

Next, let M be a general bundle of degree 1, and consider a general extension

$$0 \to E \to W \to M \to 0.$$

For each $P \in \mathcal{D}(V)$, we have

$$h^0(W \otimes P) \;\geq\; h^0(E \otimes P) \;\geq\; t - 1.$$

Hence $\mathcal{D}(W)$, if it exists, contains $(t-1) \cdot \mathcal{D}(V)$.

It is not difficult to show that $\mathcal{D}(W)$ is indeed a divisor and that W is stable. $\square$

Reducible theta divisors and families of maximal subbundles

In each of the three examples described, the bundle contains a "large" family of maximal subbundles of nongenerically high degree. (More precisely, some Quot scheme associated to the bundle, which for a generic bundle would be empty or finite, has positive dimension.)

Conjecture: Any stable bundle of integral slope with a non-integral theta divisor contains a positive-dimensional family of maximal subbundles of nongeneric degree.