On “essentially variable” variable Lebesgue space problems

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- ... remain true for variable exponents but they are unsatisfactory
- ... remain true only for *certain* variable exponents
- ... are **never** true when exponents are not constant
- ... (even!) cannot be stated when exponents are not constant

Variable Exponent Spaces: problem types

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Certain problems can be considered only in the framework of variable exponent spaces: in the constant case the problem is trivial or it does not make sense at all

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Three Examples (we are going to see)

Problems involving ...

- ... integrability properties of the exponent
- ... the decreasing rearrangement of the exponent
- ... the measure of the level sets of the exponent

Variable (Exponent) Lebesgue Spaces, $p(\cdot) < \infty$ case

$\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ (Lebesgue) measurable, $\quad 0 < |\Omega| \leq \infty$

$f : \Omega \to \mathbb{R}$ measurable

$p(\cdot) : \Omega \to [1, \infty[$ measurable

$$f \in L^{p(\cdot)}(\Omega) \overset{\text{def.}}{\Longleftrightarrow} \|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\} < \infty$$

(convention: $\inf \emptyset = \infty$)

$L^{p(\cdot)}(\Omega)$ is a Banach Function Space, **never** r.i. (unless $p(\cdot)$ is constant)

Remark :

$$|f(\cdot)|^{p(\cdot)} \in L^1(\Omega) \quad \Longrightarrow \quad f \in L^{p(\cdot)}(\Omega)$$

Some books dealing with Variable Lebesgue Spaces

- RŮŽIČKA, *Electrorheological Fluids: Modeling and Mathematical Theory,* Springer (2000)
- MESKHI, *Measure of non-compactness for integral operators in weighted Lebesgue spaces,* Nova Science Publishers (2009)
- DIENING, HARJULEHTO, HÄSTÖ, RŮŽIČKA, *Lebesgue and Sobolev spaces with Variable Exponents,* Lecture Notes in Mathematics 2017, Springer (2011)
- LANG, EDMUNDS, *Eigenvalues, embeddings and generalised trigonometric functions,* Springer (2011)
- CRUZ-URIBE, F., *Variable Lebesgue Spaces: Foundations and Harmonic Analysis,* Birkhäuser (2013)
- PICK, KUFNER, JOHN, FUČÍK, *Function spaces 1,* De Gruyter (2013)
- IZUKI, NAKAI, SAWANO, *Function Spaces with variable exponents - an introduction,* Scientiae Math. Japon. 77 (2), 276, (2014), 187–315
- EDMUNDS, LANG, MÉNDEZ, *Differential operators on spaces of variable integrability,* World Scientific (2014)

Orlicz spaces

$\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ measurable, $\quad 0 < |\Omega| < \infty$

$f : \Omega \to \mathbb{R}$ measurable

$\Phi : [0, \infty[\to [0, \infty[$ continuous, strictly increasing, convex, $\Phi(0) = 0$, $\Phi'(0) = 0$, $\Phi'(\infty) = \infty$

$$f \in L^{\Phi}(\Omega) \quad \Leftrightarrow \quad \|f\|_{L^{\Phi}(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \Phi\left(\frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\} < \infty$$

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$\Phi(t) \approx \exp(t)$, $\quad t$ large $\quad \longrightarrow \quad \mathrm{EXP}(\Omega)$

$\Phi(t) \approx \exp(t^{\beta})$, $\quad t$ large $\quad \longrightarrow \quad \mathrm{EXP}_{\beta}(\Omega)$ $(\beta > 0)$

$\Phi(t) \approx t^p \log^{\alpha p} t$, $\quad t$ large $\longrightarrow L^p(\log L)^{\alpha p}(\Omega)$ $\quad (p > 1, \alpha \in \mathbb{R}; p = 1, \alpha > 0)$

$\Phi(t) \approx \dfrac{t^p}{\log t}$, $\quad t$ large $\quad \longrightarrow \quad \dfrac{L^p}{\log L}(\Omega)$ $\quad (p > 1)$

The Hardy-Littlewood maximal operator

For $f \in L^1_{loc}(\mathbb{R}^N)$, $N \geq 1$, let M be the maximal operator defined by

$$Mf(x) = \sup_{Q \ni x} \fint_Q |f(y)| dy\,, \qquad x \in \mathbb{R}^N$$

where the sup is taken over all cubes $Q \subset \mathbb{R}^N$ that contain x and whose sides are parallel to the coordinate axes.

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A consequence of a special case of the maximal theorem

Let $\Omega \subset \mathbb{R}^N$ be bounded, let $1 < p < \infty$.

$$f \in L^p(\Omega) \Longrightarrow Mf \in L^p(\Omega)$$

An introductory remark

$\Omega \subset \mathbb{R}^N$ bounded

$f : \Omega \to \mathbb{R}$, $f \not\equiv 0$

$1 < p(\cdot) \in L^\infty(\Omega)$ (and **no more assumptions** on $p(\cdot)$!).

Question: $$f \in L^{p(\cdot)}(\Omega) \stackrel{?}{\Longrightarrow} Mf \in L^{p(\cdot)}(\Omega)$$

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$$f \in L^{p(\cdot)}(\Omega) \Longrightarrow p(\cdot)\log(Mf) \in \mathrm{EXP}(\Omega)$$

A question involving the integrability of the exponent

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$$\left.\begin{array}{l} \Omega \subset \mathbb{R}^N \text{ bounded} \\ \\ 1 \leq p(\cdot) \in L^\infty(\Omega) \\ \\ f \in L^{p(\cdot)}(\Omega) \, , \, f \not\equiv 0 \end{array}\right\} \Rightarrow p(\cdot) \log(Mf) \in \mathrm{EXP}(\Omega)$$

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Question

What happens if we weaken the assumption on $p(\cdot)$ in terms of integrability property? Do we get (at least some) exponential integrability of $p(\cdot)\log(Mf)$?

A question involving the integrability of the exponent

An example showing that

$$\left.\begin{array}{l} 1 < p(\cdot) \in L^r(0,1),\ 1 \le r < \infty \\ \\ f \in L^{p(\cdot)}(0,1)\ ,\ f \not\equiv 0,\ \beta > 0 \end{array}\right\} \not\Rightarrow p(\cdot)\log(Mf) \in \mathrm{EXP}_\beta(0,1)$$

Take $p(x) = x^{-b} + 1$, $0 < b < 1/r$ and $f(x) = (1/\sqrt{x})^{1/p(x)}$

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F. , J. Funct. Spaces (2015)

Let $\Omega \subset \mathbb{R}^N$ be bounded, $1 \le p(\cdot) < \infty$, and let $f \in L^{p(\cdot)}(\Omega)$, $f \not\equiv 0$.

- If $p(\cdot) \notin \bigcup_{\beta>0} \mathrm{EXP}_\beta(\Omega)$, then $p(\cdot)\log(Mf)$ is not necessarily in $\bigcup_{\beta>0} \mathrm{EXP}_\beta(\Omega)$

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- If $p(\cdot) \notin \bigcup_{\beta>0} \mathrm{EXP}_\beta(\Omega)$, then $p(\cdot)\log(Mf)$ is not necessarily in $\bigcup_{\beta>0} \mathrm{EXP}_\beta(\Omega)$
- If $p(\cdot) \in \bigcup_{\beta>0} \mathrm{EXP}_\beta(\Omega)$ then also $p(\cdot)\log(Mf)$ does, and in particular

$$p(\cdot) \in \mathrm{EXP}_\beta(\Omega) \Longrightarrow p(\cdot)\log(Mf) \in \mathrm{EXP}_{\beta/(\beta+1)}(\Omega)$$

The decreasing rearrangement

$\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ measurable, $\quad 0 < |\Omega| < \infty$

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$\mu_f : [0, \infty[\to [0, |\Omega|]$ is the distribution function of f, defined by

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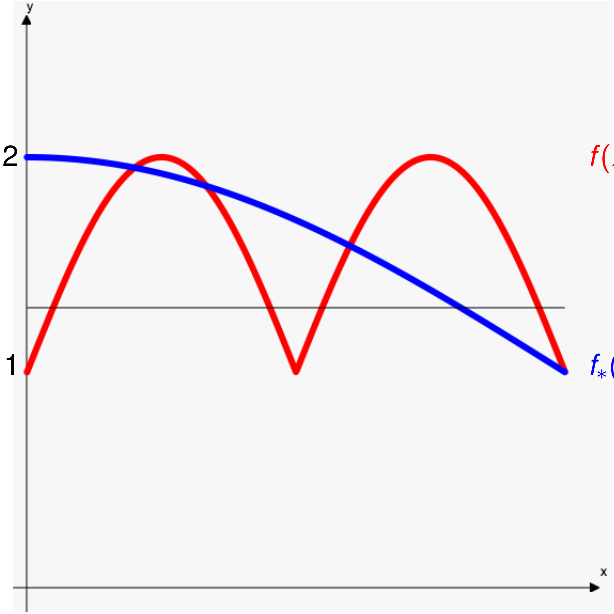
$\|f\|_{L^p(\Omega)} = \|f_*\|_{L^p(0,|\Omega|)} \qquad (1 \leq p \leq \infty)$
and more generally
$\|f\|_{L^\Phi(\Omega)} = \|f_*\|_{L^\Phi(0,|\Omega|)}$

The decreasing rearrangement: an example



$f(x) = |\sin x| + 1$

$f_*(x) = \cos\left(\frac{x}{4}\right) + 1$

Introduction to the Grand Lebesgue Spaces

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Since $L^p(\Omega) \subsetneq \bigcap_{0<\epsilon<p-1} L^{p-\varepsilon}(\Omega)$, we may consider

$$f \in \bigcap_{0<\epsilon<p-1} L^{p-\varepsilon}(\Omega) - L^p(\Omega)$$

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$L^{p)}(\Omega) =$grand L^p space$= \{f \;:\; \|f\|_{p)} < \infty\}$

[IWANIEC-SBORDONE, (1992)]

Properties of the Grand Lebesgue Spaces

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$$L^p(\Omega) \subsetneq \frac{L^p}{\log L}(\Omega) \subsetneq L^{p)}(\Omega) \subsetneq \bigcap_{\alpha>1} \frac{L^p}{\log^\alpha L(\Omega)} \subsetneq \bigcap_{0<\epsilon<p-1} L^{p-\varepsilon}(\Omega)$$

Grand Lebesgue Spaces: variants and applications

Associate space: *the small Lebesgue spaces* [F. Collect. Math. 2000], [Capone,F. J. Funct. Spaces 2005], [F., Karadzhov, Z. Anal. Anwend. 2004], [F. Rakotoson, Math. Ann. 2003], [Di Fratta, F., Nonlinear Anal. 2009], [Cobos, Kühn, Calc. Var. PDE 2014], [Anatriello, Collect. Math. 2014], *$G\Gamma$ spaces* [F., Rakotoson, JMAA 2008], [F., Rakotoson, Zitouni, Indiana Univ. Math. J. 2009], [Gogatishvili, Pick, Soudský, Studia Math. 2014]

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Applications*: Integrability of the Jacobian* [IWANIEC-SBORDONE, ARMA 1992], [GRECO, Diff. Int. Eq. 1993] *Mappings of finite distortion* [IWANIEC, KOSKELA, ONNINEN Invent. Math. 2001], *PDEs* [SBORDONE, Matematiche, 1996], [F.,SBORDONE, Studia Math. 1998], [GRECO, IWANIEC, SBORDONE, Manuscripta Math. 1997], [BOCCARDO, CRAS 1997], [F.,MERCALDO,RAKOTOSON Discr. Cont. Dyn. Systems 2002], *Calculus of Var.* [F.,RAKOTOSON, Calc.Var. 2005], *Extrapolation* [CAPONE, F., KRBEC, J. Ineq. Appl. 2006]

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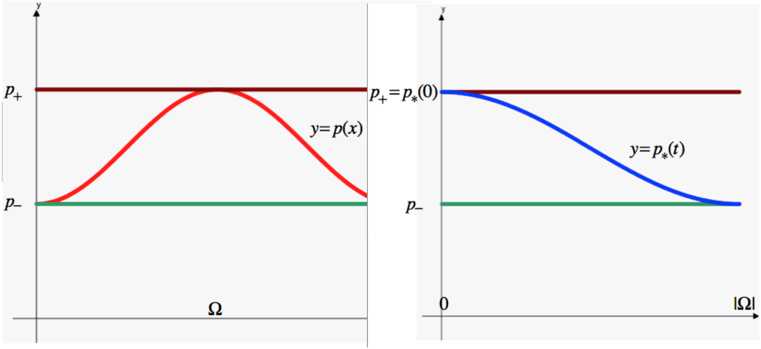
Sobolev-type results: [Fusco, Lions, Sbordone, Proc. AMS 1996], [F.,Krbec,Schmeisser, J. Funct. Anal. 2014], [Maeda,Mizuta,Ohno,Shimomura, Ann.Acad.Sci.Fenn. 2015], [Futamura, Mizuta, Ohno, Proc. Int. Symp. on Banach and Funct. Spaces IV, 2012]

Rearranging exponents: first elementary embeddings

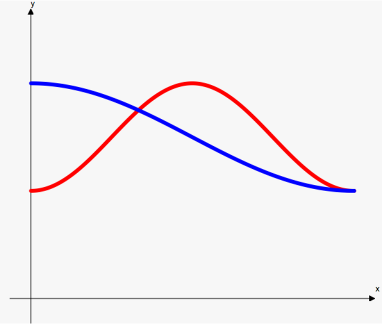
$$p_- = \operatorname{ess\,inf} p = \operatorname{ess\,inf} p_* \quad , \quad p_+ = \operatorname{ess\,sup} p = \operatorname{ess\,sup} p_*$$



$$L^{p_+}(\Omega) \subseteq L^{p(\cdot)}(\Omega) \subseteq L^{p_-}(\Omega) \qquad L^{p_+}(0,|\Omega|) \subseteq L^{p_*(\cdot)}(0,|\Omega|) \subseteq L^{p_-}(0,|\Omega|)$$

$L^{p(\cdot)}(0,1) \neq L^{p_*(\cdot)}(0,1)$, unless $p(\cdot) \searrow$

$(\textit{in the picture}: \; p(x) = 1 + \sin^2 x \quad p_*(x) = 1 + \cos^2(x/2), \; x \in (0,\pi))$



Proposition: $L^{p(\cdot)}(0,1)$ *and* $L^{p_*(\cdot)}(0,1)$ *are never comparable, unless* $p(\cdot) = p_*(\cdot)$ *a.e., i.e.* $p(\cdot) \searrow$

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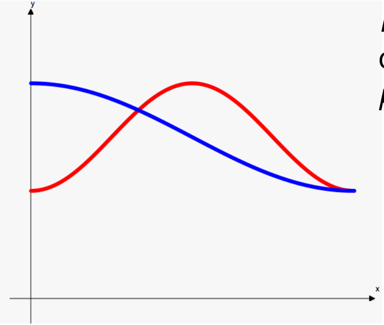

Proposition: *$L^{p(\cdot)}(0,1)$ and $L^{p_*(\cdot)}(0,1)$ are never comparable, unless $p(\cdot) = p_*(\cdot)$ a.e., i.e. $p(\cdot) \searrow$*

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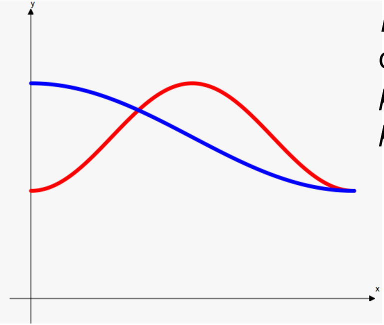

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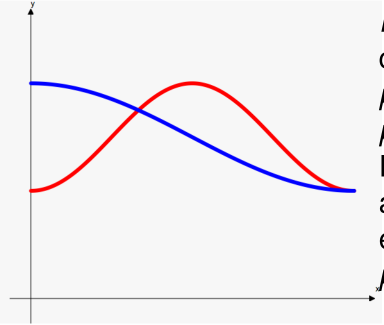

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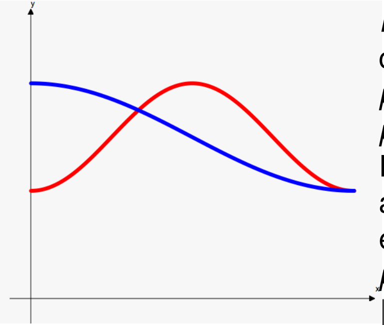

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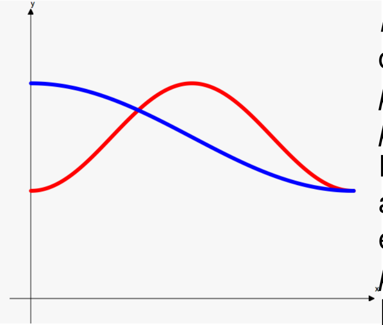

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Conclusion: they are both true, i.e. $p(\cdot) = p_*(\cdot)$ a.e.

Some finer inclusions

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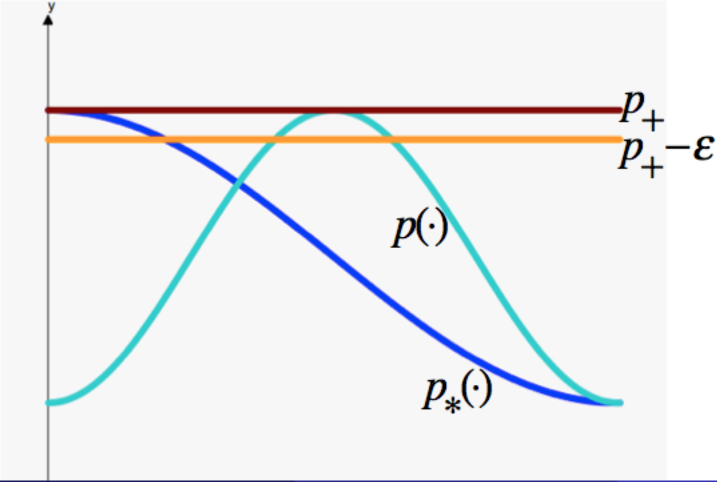
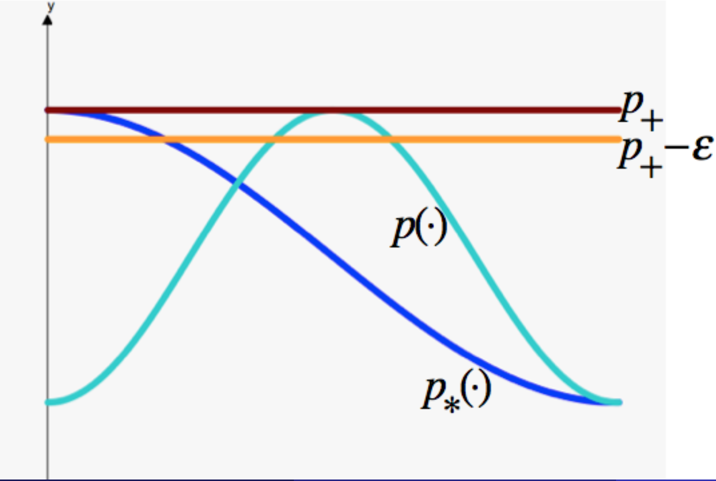


Some finer inclusions

$$L^{p_+}(\Omega) = \{f : f_* \in L^{p_+}(0, |\Omega|)\} \subseteq \{f : f_* \in L^{p_*(\cdot)}(0, |\Omega|)\} \subseteq L^{p_+ - \varepsilon}(\Omega)$$

Last inclusion follows from $L^{p_+ - \varepsilon}(\Omega) = \{f : f_* \in L^{p_+ - \varepsilon}(0, |\Omega|)\}$



Grand Lebesgue spaces come into play!

$$L^{p_+}(\Omega) \subseteq \{f \,:\, f_* \in L^{p_*(\cdot)}(0, |\Omega|)\} \subseteq \bigcap_{0<\epsilon<p-1} L^{p_+-\varepsilon}(\Omega)$$

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A result involving the rearrangement of the exponent

F., Rakotoson, Sbordone, Comm. Contemp. Math. (2015)

If $1 < p_- \le p(\cdot) \le p_+ < \infty$ in $(0,1)$, $p_*(0) > 1$, and

$$(LH_{t=0}) \qquad \limsup_{t\to 0} |p_*(t) - p_*(0)||\log t| < \infty$$

then $\{f : f_* \in L^{p_*(\cdot)}(0, |\Omega|)\}$ coincides with the Banach function space $L^{p_*(\cdot)}_{**}(\Omega)$ defined through the norm

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and

$$L^{p_+}(\Omega) \subseteq L_{**}^{p_*(\cdot)}(\Omega) = \{f \; : f_* \in L^{p_*(\cdot)}(0,|\Omega|)\} \subset L^{p_+)}(\Omega) = L^{p_*(0))}(\Omega)$$

last inclusion being strict.

Approximate identities in L^p, $1 \leq p \leq \infty$

Let $\varphi(x) = \begin{cases} \dfrac{1}{\int_{|x|\leq 1} e^{\frac{1}{|x|^2-1}}\,dx} e^{\frac{1}{|x|^2-1}} & \text{if} |x| < 1 \\ 0 & \text{if} |x| \geq 1 \end{cases}$ and let $\varphi_t(x) = t^{-N}\varphi\left(\frac{x}{t}\right), t > 0$

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Moreover, if the radial and decreasing function (as $|x|$ decreases) $\Phi(x) = \sup_{|y|\ge|x|} |\varphi(y)| \in L^1(\mathbb{R}^N)$ (as in the example above, which is such that $\Phi = \varphi$), $\{\varphi_t \ : \ t > 0\}$ is called **potential type approximation identity** and $\varphi_t * f \to f$ a.e. as $t \to 0$ (this is true for $p = \infty$, too).

Approximate identities in $L^{p(\cdot)}$, norm convergence

Theorem [Diening (2004)], [Cruz-Uribe, F. (2007,2013)]

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Approximate identities in $L^{p(\cdot)}$, converg. in measure

The norm convergence result for approximate identities is obtained for special exponents p such that $p_+ < \infty$ and examples exist such that the norm convergence does not hold when $p_+ = \infty$ (we will see one).

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Cruz-Uribe, F., Analysis and Applications (2015)

If $\{\varphi_t : t > 0\}$ is a potential type approximation identity, then $\forall f \in L^{p(\cdot)}(\mathbb{R}^N)$

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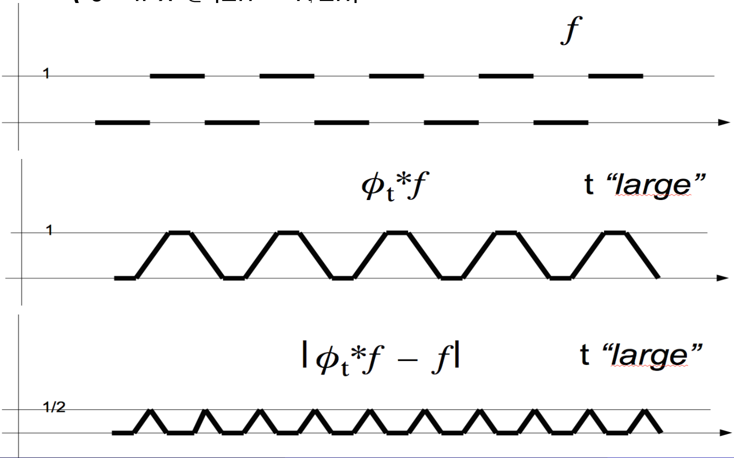
In [Cruz-Uribe, F. (2013)] we showed $p_+ < \infty \Rightarrow \varphi_t * f \to f$ in measure and we gave the following example of $p(\cdot)$ such that $p_+ = \infty$ and $\varphi_t * f \not\to f$ in measure (and therefore in norm). In view of the recent result, we will give a new comment on it.

Example: $p_+ = \infty$ and $\varphi_t * f \not\to f \in L^{p(\cdot)}(\mathbb{R})$ in measure

Let $p(x) = 1 + |x|$, $x \in \mathbb{R}$, $\phi(x) = \chi_{(-1/2,1/2)}$, and let

$$f(x) = \begin{cases} 1 & \text{if } x \in [2n, 2n+1] \\ 0 & \text{if } x \in]2n-1, 2n[\end{cases}, \quad n \geq 1 \qquad f \in L^{p(\cdot)}(\mathbb{R})$$



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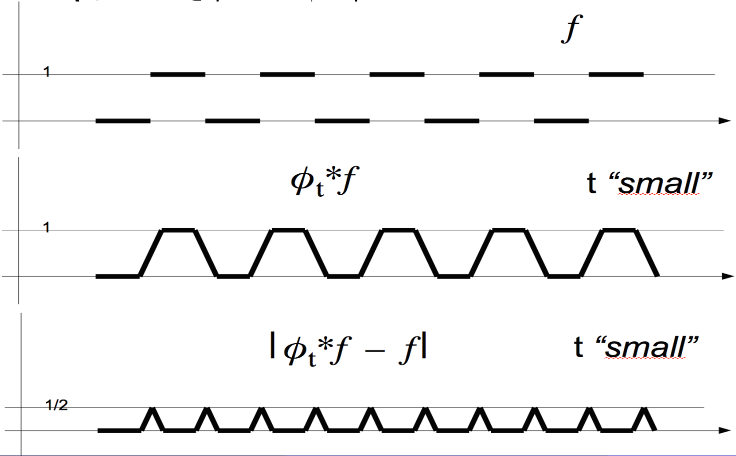

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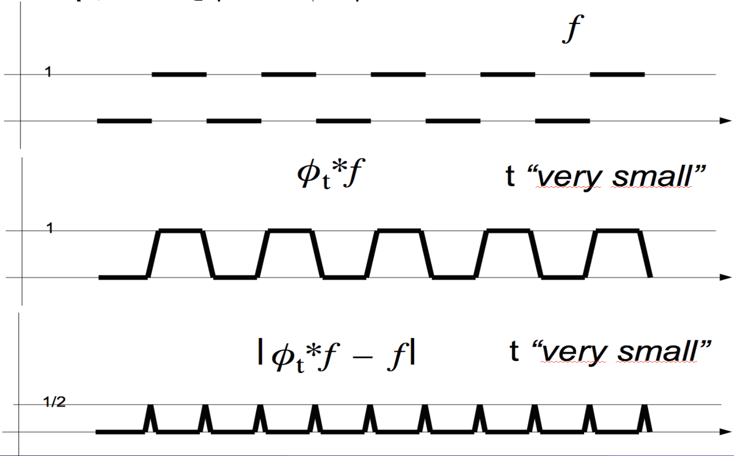

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thank you!