A singular limit problem for viscous compressible fluids in presence of capillarity

Francesco Fanelli

Centro di Ricerca Matematica "Ennio De Giorgi"

SCUOLA NORMALE SUPERIORE

AMS – EMS – SPM INTERNATIONAL MEETING 2015

Special session "PDEs: ambitious Mathematics for real-life applications"

Porto – June 10-13, 2015

Contents of the talk

COMPRESSIBLE FLUIDS WITH CAPILLARITY EFFECTS

The general system

$$
\begin{cases}
\partial_t \rho \, + \, \mathrm{div}\, (\rho u) \, = \, 0 \\
\partial_t \, (\rho u) \, + \, \mathrm{div}\, \big(\rho u \otimes u\big) \, + \, \nabla \Pi(\rho) \, = \\
\qquad\qquad = \, \mathrm{div}\, \big(\nu(\rho) D u + \lambda(\rho) \mathrm{div}\, u \, \mathrm{Id}\,\big) \, + \, \kappa \rho \nabla \big(\sigma'(\rho) \Delta \sigma(\rho)\big)
\end{cases}
$$

- $\rho(t,x) \, \geq \, 0$ density of the fluid
- $u(t,x) \, \in \, \mathbb{R}^3$ velocity field
- $\Pi(\rho) \, = \, \rho^\gamma \, / \, \gamma$ pressure of the fluid $(\gamma \geq 1)$
- $Du \, := \, (1/2) \, \big(\nabla u \, + \, {}^t \nabla u\big)$
- $\kappa \, > \, 0$ capillarity coefficient

- $\kappa = 0\,, \quad \nu(\rho) = \nu > 0\,, \quad \lambda(\rho) = \lambda\,, \quad \nu + \lambda > 0$

$\Longrightarrow$ existence of global weak solutions

(P.-L. Lions – 1993)

- $\kappa > 0\,, \quad \sigma(\rho) = \rho\,, \quad \nu > 0\,, \quad \lambda = \nu/3$

$\Longrightarrow$ local existence of strong solutions

global if initial data close to a stable equilibrium

(Hattori & Li – 1996)

(Danchin & Desjardins – 2001)

Navier-Stokes-Korteweg system

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho u) = 0 \\ \partial_t\,(\rho u) + \mathrm{div}\big(\rho u \otimes u\big) + \nabla\Pi(\rho) - \nu_0\,\mathrm{div}\,\big(\rho\,Du\big) - \kappa\,\rho\nabla\Delta\rho = 0 \end{cases}$$

- Capillarity term:

$$\kappa > 0 \qquad \text{and} \qquad \sigma(\rho) = \rho$$

- Viscosity cofficients:

$$\nu(\rho) = \nu_0\rho \qquad \text{and} \qquad \lambda(\rho) \equiv 0$$

▷ Degeneracy for $\rho \sim 0$

▷ Capillarity $\rightsquigarrow$ control on $\nabla^2\rho$

Theorem (Bresch & Desjardins & Lin – 2003)

$\exists$ global in time "weak" solutions (ρ, u)

Remarks

(i) Weak solutions *à la Leray*

(ii) "Weak": momentum equation tested on $\rho\varphi$, $\varphi \in \mathcal{D}(\Omega)$

$$\int_0^T \int_\Omega \Big(\rho^2 u \cdot \partial_t \varphi + \rho^2 u \otimes u : \nabla\varphi - \rho^2 u \cdot \varphi \mathrm{div} u -$$

$$-\nu\rho^2 D(u) : \nabla\varphi - \nu\rho D(u) : \varphi \otimes \nabla\rho + \Pi(\rho)\rho \mathrm{div}\varphi -$$

$$-\kappa\rho^2 \Delta\rho \mathrm{div}\varphi - 2\kappa\rho\nabla\rho \cdot \varphi\Delta\rho \Big) dxdt = - \int_\Omega \rho_0^2 u_0 \cdot \varphi(0) dx$$

On the proof

1) A priori estimates

▷ Classical energy

$$\Longrightarrow \quad \rho \in L^\infty_T L^\gamma \,,\ \nabla\rho \in L^\infty_T L^2 \,,\ \sqrt{\rho}\, u \in L^\infty_T L^2 \,,\ \sqrt{\rho}\, Du \in L^2_T L^2$$

▷ *BD entropy* conservation

$$\Longrightarrow \quad \nabla^2\rho \in L^2_T L^2 \,,\ \nabla\sqrt{\rho} \in L^\infty_T L^2$$

2) Construction of smooth approximated solutions $\big(\rho_n \,,\, u_n\big)_n$

3) Stability analysis

NAVIER-STOKES-KORTEWEG

WITH

CORIOLIS FORCE

Fluid models with Coriolis force

- *Motivation*: description of large scale phenomena
  - ▷ quantitative aspects
  - ▷ qualitative aspects (physical effects)

- *General hypotheses*:
  - (i) Rotation around the vertical axis x^3
  - (ii) Constant rotation speed

  $\Longrightarrow$ rotation operator: $u \mapsto \left(e^3 \times u\right) / Ro$

  - (iii) Complete slip boundary conditions

  $\Longrightarrow$ NO boundary layers effects

- *Singular perturbation problem*: $Ro \sim \varepsilon$

$\Longrightarrow$ asymptotic behavior of weak solutions for $\varepsilon \to 0$

N-S-K with Coriolis force

$$\begin{cases} \partial_t \rho + \operatorname{div}\left(\rho\, u\right) = 0 \\ \partial_t\left(\rho\, u\right) + \operatorname{div}\left(\rho\, u \otimes u\right) + \dfrac{1}{\varepsilon^2}\nabla\Pi(\rho) + \\ \qquad + \dfrac{e^3 \times \rho\, u}{\varepsilon} - \nu \operatorname{div}\left(\rho\, Du\right) - \dfrac{1}{\varepsilon^{2(1-\alpha)}}\, \rho\, \nabla\Delta\rho = 0 \end{cases}$$

- $\Omega = \mathbb{R}^2 \times]0,1[$ + complete slip boundary conditions
- $\Pi(\rho) = \rho^2 / 2$
- Mach number $\sim \varepsilon$ and Rossby number $\sim \varepsilon$
- $\kappa \sim \varepsilon^{2\alpha}$, with $0 \leq \alpha \leq 1$

▷ *Ill-prepared* initial data

(i) $\rho_{0,\varepsilon} = 1 + \varepsilon\, r_{0,\varepsilon}$, with $\left(r_{0,\varepsilon}\right)_\varepsilon \subset H^1(\Omega) \cap L^\infty(\Omega)$

(ii) $\left(u_{0,\varepsilon}\right)_\varepsilon \subset L^2(\Omega)$

Statements

- *Vanishing capillarity* limit: $0 < \alpha \leq 1$

Theorem (F. – 2014)

$\left(\rho_\varepsilon\,,\,u_\varepsilon\right)_\varepsilon$ weak solutions, $\rho_\varepsilon \,=\, 1 \,+\, \varepsilon\, r_\varepsilon$

$r_\varepsilon \rightharpoonup r\,,\qquad \sqrt{\rho_\varepsilon}\, u_\varepsilon \rightharpoonup u$

a) $\operatorname{div} u \,\equiv\, 0$

b) $u \,=\, \left(u^h(x^h)\,,\,0\right)$, with $u^h \,=\, \nabla_h^\perp r$

c) r solves a quasi-geostrophic equation

$$\partial_t\left(r \,-\, \Delta_h r\right) \,+\, \nabla_h^\perp r \cdot \nabla_h \Delta_h r \,+\, \nu\, \Delta_h^2 r \,=\, 0$$

- *Constant capillarity* regime: $\alpha = 0$

Theorem (F. – 2014)

$\left(\rho_\varepsilon\,,\,u_\varepsilon\right)_\varepsilon$ weak solutions, $\rho_\varepsilon = 1 + \varepsilon\, r_\varepsilon$

$r_\varepsilon \rightharpoonup r\,, \qquad \sqrt{\rho_\varepsilon}\,u_\varepsilon \rightharpoonup u$

a) $\operatorname{div} u \equiv 0$

b) $u = \left(u^h(x^h)\,,\,0\right),$ with $u^h = \nabla_h^\perp\left(\mathrm{Id} - \Delta_h\right)r$

c) r solves

$$\partial_t\Big((\mathrm{Id} - \Delta_h + \Delta_h^2)r\Big) + \\ + \nabla_h^\perp\left(\mathrm{Id} - \Delta_h\right)r \cdot \nabla_h\Delta_h^2 r + \nu\Delta_h^2\left(\mathrm{Id} - \Delta_h\right)r = 0$$

Related results

- 2-D viscous shallow water with friction terms
  (Bresch & Desjardins – 2003)
  - Viscosity $= -\nu\,\mathrm{div}\big(\rho\,Du\big)$, capillarity $= -\rho\nabla\Delta\rho$

- General Navier-Stokes-Korteweg system
  (Jüngel & Lin & Wu – 2014)
  - Viscous tensor $\rightsquigarrow$ $-\,\mathrm{div}\big(\nu(\rho)\,Du\big)$
  - Capillarity term $\rightsquigarrow$ $-\kappa\,\rho\,\nabla\big(\sigma'(\rho)\,\Delta\sigma(\rho)\big)$
  - Strong solutions framework; local in time study

▷ Incompressible + high rotation + *vanishing capillarity*

▷ $\Omega = \mathbb{T}^2$

▷ Well-prepared initial data

Main steps of the proof

(i) Uniform bounds

- ▷ Classical energy conservation $+$ BD entropy
- ! Control of the rotation term *uniformly* in ε
  - $\rightarrow$ Necessary to have $\|\rho_\varepsilon - 1\|_{L^\infty_T L^2} \sim O(\varepsilon)$

(ii) Constraint on the limit

- ▷ Taylor-Proudman theorem $+$ stream-function relation

(iii) Propagation of acoustic waves

- ▷ Spectral analysis of the singular operator $+$ RAGE Theorem
  (Feireisl & Gallagher & Novotný – 2012)
- ! Microlocal symmetrization argument
- ! $0 < \alpha < 1 \quad \Longrightarrow \quad$ anisotropy of scaling

Variable rotation axis

- Coriolis operator $\rightsquigarrow$ $\mathfrak{C}(\rho, u) = \mathfrak{c}(x^h)\, e^3 \times \rho\, u$
  - (i) $\mathfrak{c}$ has non-degenerate critical points
  - (ii) $\nabla_h \mathfrak{c} \in \mathcal{C}_\mu(\mathbb{R}^2)$ for $\mu =$ admissible modulus of continuity

Theorem (F. – 2015)

$\big(\rho_\varepsilon\,,\, u_\varepsilon\big)_\varepsilon$ weak solutions, $\rho_\varepsilon = 1 + \varepsilon\, r_\varepsilon$

$r_\varepsilon \rightharpoonup r\,,\qquad \sqrt{\rho_\varepsilon}\, u_\varepsilon \rightharpoonup u$

a) $\mathrm{div}\, u \equiv 0$

b) $u = \big(u^h(x^h)\,,\, 0\big)\,,$ with $\mathfrak{c}(x^h)\, u^h = \nabla_h^\perp \big(\mathrm{Id} - \Delta_h\big) r$

c) r solves a *linear* “parabolic” equation

Remarks

1) Singular perturbation operator: *variable coefficients*

$\Longrightarrow$ compensated compactness arguments

- **Gallagher & L. Saint-Raymond – 2006**
- **Feireisl & Gallagher & Gérard-Varet & Novotný – 2012**

2) Novelties:

- Surface tension term
- Less regularity available for the approximation

3) Regularity of $\mathfrak{c}(x^h)$

- Zygmund conditions

THANK YOU !