Cohomological dimensions of Hopf algebras

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AMS-EMS-SPM Meeting
Porto, June 2015
Session Algebraic and Categorical Aspects of Hopf algebras

Question 1

Let A, B be Hopf algebras such that

$$\mathrm{Comod}(A) \simeq^{\otimes} \mathrm{Comod}(B)$$

How are their Hochschild cohomologies related ? In particular do we have $\mathrm{cd}(A) = \mathrm{cd}(B)$?

$\mathrm{cd}(A)$ is the usual Hochschild cohomological dimension :

$$\mathrm{cd}(A) = \sup\{n : \ H^n(A, M) \neq 0 \text{ for some } A-\text{bimodule } M\} \in \mathbb{N} \cup \{\infty\}$$

Strategy

(1) Consider a cohomology theory that is well-behaved with respect to this situation : Gerstenhaber-Schack cohomology.
(2) Try to compare $H^*_{\mathrm{GS}}(A, -)$ and $H^*(A, -)$.

Yetter-Drinfeld modules

A is a Hopf algebra over a field k.

Definition

A (right-right) Yetter-Drinfeld module over A is a right A-comodule and right A-module V satisfying the condition, $\forall v \in V$, $\forall a \in A$,

$$(v \leftarrow a)_{(0)} \otimes (v \leftarrow a)_{(1)} = v_{(0)} \leftarrow a_{(2)} \otimes S(a_{(1)})v_{(1)}a_{(3)}$$

$\rightsquigarrow$ category $\mathcal{YD}_A^A$, with $\mathcal{YD}_A^A \simeq^{\otimes} \mathcal{Z}(\mathrm{Comod}(A)) \simeq^{\otimes} \mathcal{Z}(\mathrm{Mod}(A))$

Examples and definition

(1) $A_{\mathrm{coad}} := A_A$ as a right A-module, and $\mathrm{ad}_r(a) = a_{(2)} \otimes S(a_{(1)})a_{(3)}$.
(2) More generally, if $V \in \mathrm{Comod}(A) \rightsquigarrow V \boxtimes A$, with

$$(v \otimes a) \leftarrow b = v \otimes ab, \ \alpha_{V \boxtimes A}(v \otimes a) = v_{(0)} \otimes a_{(2)} \otimes S(a_{(1)})v_{(1)}a_{(3)}$$

In particular $k \boxtimes A = A_{\mathrm{coad}}$. A Yetter-Drinfeld module of type $V \boxtimes A$ is called *free*. A *relative projective* Yetter-Drinfeld module is a direct summand of a free one. These are projective as A-modules.

Gerstenhaber-Schack cohomology

$V \in \mathcal{YD}_A^A \rightsquigarrow H^*_{\mathrm{GS}}(A, V)$, Gerstenhaber-Schack cohomology, defined by an explicit bicomplex (originally defined in terms of Hopf bimodules).
$H^*_{\mathrm{GS}}(A, k) = H^*_b(A)$ is the bialgebra cohomology of A.

This cohomology theory was introduced in view of deformation theory. It was later used by Stefan to show that the set of isomorphism classes of semisimple cosemisimple Hopf algebras of a given dimension is finite.

Theorem (Taillefer)

$$H^*_{\mathrm{GS}}(A, V) \simeq \mathrm{Ext}^*_{\mathcal{YD}_A^A}(k, V)$$

We will use this description as a definition.

Theorem

Let $F : \mathrm{Comod}(A) \simeq^{\otimes} \mathrm{Comod}(B)$ *be a tensor equivalence. Then* F *extends to a tensor equivalence* $\hat{F} : \mathcal{YD}_A^A \simeq^{\otimes} \mathcal{YD}_B^B$ *such that*

1. *For any* $V \in \mathcal{YD}_A^A$, $H^*_{\mathrm{GS}}(A, V) \simeq H^*_{\mathrm{GS}}(B, \hat{F}(V))$. *In particular*

$$\mathrm{cd}_{\mathrm{GS}}(A) = \mathrm{cd}_{\mathrm{GS}}(B)$$

2. $V \in \mathcal{YD}_A^A$ *is relative projective* $\Rightarrow \hat{F}(V) \in \mathcal{YD}_B^B$ *is relative projective. In particular, if* $\mathbf{P}_{.} \to k$ *is a resolution of* k *by relative projectives in* $\mathcal{YD}_A^A$, *then* $\hat{F}(\mathbf{P}.) \to k$ *is a resolution of* k *by relative projectives in* $\mathcal{YD}_B^B$.

This result shows that the Hochschild cohomologies of A and B are indeed related.
(recall that $H^*(A, M) \simeq \mathrm{Ext}^*_A(k_\varepsilon, M')$, where if M is an A-bimodule, M' is the right A-module defined by $m \leftarrow a = S(a_{(1)}) \cdot m \cdot a_{(2)}$)

Recall that a Hopf algebra A is said to be co-Frobenius if $\mathrm{Comod}(A)$ has enough projectives. This implies that $\mathcal{YD}_A^A$ also has enough projectives.

Theorem

Let A be co-Frobenius. If $\mathbf{P}_.$ is a resolution of k by relative projectives in $\mathcal{YD}_A^A$, then

$$H^*_{\mathrm{GS}}(A, V) \simeq H^*(\mathrm{Hom}_{\mathcal{YD}_A^A}(\mathbf{P}_., V))$$

for any $V \in \mathcal{YD}_A^A$.

This follows from the combination of results by Shnider/Sternberg and the previous one by Taillefer.

The previous result enables us to use standard resolutions, an provides an explicit complex to compute Gerstenhaber-Schack cohomology. This leads to

Theorem

Let A be co-Frobenius. There exists a functor

$$\begin{aligned} \mathrm{Bimod}(A) &\longrightarrow \mathcal{YD}_A^A \\ M &\longmapsto M\#A \end{aligned}$$

such that

$$H^*(A, M) \simeq H^*_{\mathrm{GS}}(A, M\#A)$$

In particular we have $\mathrm{cd}(A) \leq \mathrm{cd}_{\mathrm{GS}}(A)$.

$M\#A$ is $M \otimes A$ endowed with the Yetter-Drinfeld module structure defined by $m \otimes a \mapsto m \otimes a_{(1)} \otimes a_{(2)}$, $(m \otimes a) \leftarrow b = S(b_{(2)}).m.b_{(3)} \otimes S(b_{(1)})ab_{(4)}$.

This shows that the Gerstenhaber-Schack cohomology of a co-Frobenius Hopf algebra completely determines its Hochschild cohomology.

Back to Question 1, our most general answer is

Corollary

Let A and B be co-Frobenius Hopf algebras such that $\mathrm{Comod}(A) \simeq^{\otimes} \mathrm{Comod}(B)$. *Then there exists two functors*

$$F_1 : \mathrm{Bimod}(A) \to \mathcal{YD}_B^B \quad \text{and} \quad F_2 : \mathrm{Bimod}(B) \to \mathcal{YD}_A^A$$

such that for any A-bimodule M and any B-bimodule N,

$$H^*(A, M) \simeq H^*_{\mathrm{GS}}(B, F_1(M)) \quad \text{and} \quad H^*(B, N) \simeq H^*_{\mathrm{GS}}(A, F_2(N))$$

In particular we have $\max(\mathrm{cd}(A), \mathrm{cd}(B)) \leq \mathrm{cd}_{\mathrm{GS}}(A) = \mathrm{cd}_{\mathrm{GS}}(B)$.

This leads to a new question :

Question 2

Let A be co-Frobenius. Assume that $\mathrm{car}(k) = 0$. Is it true that $\mathrm{cd}(A) = \mathrm{cd}_{\mathrm{GS}}(A)$?

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Let A be co-Frobenius. Assume that $\mathrm{car}(k) = 0$. Is it true that $\mathrm{cd}(A) = \mathrm{cd}_{\mathrm{GS}}(A)$?

- Positive answer to Question 2 $\Rightarrow$ positive answer to the last part of Question 1.
- True for $A = k\Gamma$, A finite-dimensional (both dimensions are either 0 in the cosemisimple case by Larson-Radford, or ∞), and probably for $A = \mathcal{O}(G)$, with G connected reductive algebraic group.
- A positive answer to Question 2 would be a natural infinite-dimensional generalization of a famous result by Larson-Radford : a finite-dimensional cosemisimple Hopf algebra is semisimple.

Examples ($\mathrm{car}(k) = 0$)

Theorem

Let $A = \mathcal{O}_q(\mathrm{SL}_2)$ (A is co-Frobenius).

1. *k has a length 3 resolution by free Yetter-Drinfeld modules.*
2. $\mathrm{cd}(A) = 3 = \mathrm{cd}_{\mathrm{GS}}(A)$.
3. *We have*

$$H_b^n(A) \simeq \begin{cases} 0 & \text{if } n \neq 0,3 \\ k & \text{if } n = 0,3 \end{cases}$$

(Of course that $\mathrm{cd}(A) = 3$ was known since a long time)

Let $E \in \mathrm{GL}_n(k)$, $n \geq 2$, and consider the algebra $\mathcal{B}(E)$ presented by generators $(u_{ij})_{1\leq i,j\leq n}$ and relations

$$E^{-1}u^tEu = I_n = uE^{-1}u^tE,$$

where u is the matrix $(u_{ij})_{1\leq i,j\leq n}$. It has a Hopf algebra structure defined by

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj},\ \varepsilon(u_{ij}) = \delta_{ij},\ S(u) = E^{-1}u^tE$$

The Hopf algebra $\mathcal{B}(E)$, introduced by Dubois-Violette and Launer, represents the quantum symmetry group of the bilinear form associated to the matrix E. For the matrix

$$E_q = \begin{pmatrix} 0 & 1 \\ -q^{-1} & 0 \end{pmatrix} \in \mathrm{GL}_2(k)$$

we have $\mathcal{B}(E_q) = \mathcal{O}_q(\mathrm{SL}_2)$.
For $q \in k^*$ satisfying $\mathrm{tr}(E^{-1}E^t) = -q - q^{-1}$, the tensor categories of comodules over $\mathcal{B}(E)$ and $\mathcal{O}_q(\mathrm{SL}_2)$ are equivalent.

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Let $A = \mathcal{B}(E)$ (A is co-Frobenius).

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Theorem

Let $A = \mathcal{O}_q(\mathrm{PSL}_2)$ *(A is co-Frobenius). Assume that* $q + q^{-1} \neq 0$.

1. k *has a length* 3 *resolution by relative projective Yetter-Drinfeld modules.*
2. $\mathrm{cd}(A) = 3 = \mathrm{cd}_{\mathrm{GS}}(A)$.
3. *We have*

$$H_b^n(A) \simeq \begin{cases} 0 & \text{if } n \neq 0, 3 \\ k & \text{if } n = 0, 3 \end{cases}$$

Let $A_s(n)$ be the algebra presented by generators $(u_{ij})_{1 \leq i,j \leq n}$ and relations

$$\sum_k u_{ki} = 1 = \sum_k u_{ik}, \ u_{ik}u_{ij} = \delta_{kj}u_{ij}, \ u_{ki}u_{ji} = \delta_{jk}u_{ji}$$

It has a natural Hopf algebra structure and represents the quantum permutation group S_n^+ (Wang).

Theorem

Let $A = A_s(n)$ (A is co-Frobenius). Assume that $n \geq 4$.

1. *k has a length 3 resolution by relative projective Yetter-Drinfeld modules.*
2. $\mathrm{cd}(A) \leq 3 = \mathrm{cd}_{\mathrm{GS}}(A)$.
3. *We have*

$$H_b^n(A) \simeq \begin{cases} 0 & \text{if } n \neq 0, 3 \\ k & \text{if } n = 0, 3 \end{cases}$$

References

The talk was based on the following two papers :

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