Harmonic maps, the Xanthopoulos Conjecture, & Analysis of Singularities in General Relativity

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Conjecture

(Xanthopoulos, 1984) For systems of $n > 1$ PDEs for n unknown scalar fields of 2 independent variables which can be described as a harmonic mapping of (Riemannian) manifolds, complete integrability implies Lax integrability.

“A Geometric Notion of Integrability” Phys. Lett. A, v105(7) 334–338, 1984

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2 Theorems (as a possible first step) to proving the Conjecture

Implications in the analysis of singularities in GR

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Motivating Example

Consider the (1+1-dimensional) KdV Equation

$$u_t - 6uu_x + u_{xxx} = 0.$$

- Prototype for an exactly solvable nonlinear evolution equation: exhibits soliton solutions & infinite integrals of motion, arises from a least action principle, has nice analyticity/smoothness properties, etc.
- Can be reformulated as a compatibility condition for a pair of *linear* differential operators (Lax '68):

$$L_t - [L, B] = 0,$$

  where $L = -\frac{d^2}{dx^2} + u$ and $B = 4\frac{d^3}{dx^3} - 6u\frac{d}{dx} - 3u_x$.

L, B called a Lax Pair for the PDE (so KdV is called Lax-integrable).

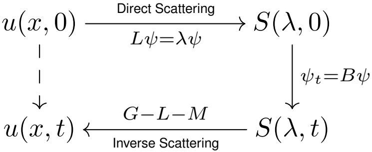
What does Lax-integrability give us?

It is intimately connected to the Inverse Scattering Mechanism (ISM)

ISM in a Nutshell for $u_t = F(u, u_x, u_{xx}, \ldots)$

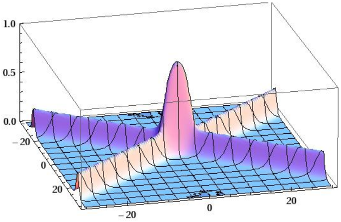
(e.g., KdV $u_t - 6uu_x + u_{xxx} = 0$)

- Assume $u(x,t) \to 0$ at infinity and let $u(x,0)$ be Cauchy data.
- Examine the associated *direct scattering problem*, $L\psi = \lambda\psi$ (e.g., $L = -\frac{d^2}{dx^2} + u(x,t)$), consisting of finding full set of scattering data $S(\lambda, 0)$. (e.g., produced by potential $u(x,0)$)
- Determine the equation of evolution for the scattering data $S(\lambda,0) \to S(\lambda,t)$. (e.g., $B = 4\frac{d^3}{dx^3} - 6u\frac{d}{dx} - 3u_x$)
- Reconstruct the potential $u(x,t)$ via scattering data $S(\lambda,t)$.



Why is this important?

- KP $(u_t + u_x u + \epsilon^2 u_{xxx})_x \pm u_{yy} = 0$
- Liouville $\triangle u = u_{xx} + u_{tt} = 0$
- sine-Gordon $\Box u = \sin u$ (sinh-Poisson $\triangle u = \sinh u$)
- cubic NLS $i\psi_t + \frac{1}{2}\psi_x^2 = K|\psi|^2\psi$
- SPE $u_{xt} = u + \frac{1}{6}(u^3)_x$
- (SAS) Einstein Vacuum, Einstein Maxwell, SDYM, ...



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(Xanthopoulos, 1984) For systems of $n > 1$ PDEs for n unknown scalar fields of 2 independent variables which can be described as a harmonic mapping of (Riemannian) manifolds, complete integrability implies Lax integrability.

Gel'fand, Levitan, Marchenko ('51/'55), Gardner, Green, Kruskal, Miura ('67), Lax ('68), Kadomsev, Petviashvili ('70), Zakharov, Mihailov ('74), Zakharov, Belinski ('78/'79), Shabat, Faddeev, Alekseev ('80s), Ablowitz, Kaup, Newell, Segur ('80s), Novikov, Manakov, Pitaevskii, Zakharov ('84), Newell ('85), Takhtadjan ('80s/'90s), Ablowitz, Clarkson ('91), Miwa, Jimbo, Date ('00), Hirota ('04), Neugebauer, Kramer, Meinel ('90s/'00s) . . .

Conjecture

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A harmonic map $f : (M^m, \mathbf{g}) \to (N^n, \mathbf{h})$ is a critical point of the Dirichlet Energy (Eells & Sampson ‘64, Sanchez ‘82):

$$\mathcal{E}[f, \mathcal{D}] = \int_{\mathcal{D}} \frac{1}{2} \mathrm{tr}_{\mathbf{g}} f^* \mathbf{h}.$$

- M = manifold of coordinates, N = manifold of fields
- Using local coordinate chart on N of the form $\{f^A : A = 1, 2, \ldots, n\}$, the Euler-Lagrange equations become

$$g^{ab} \nabla_a \nabla_b f^A + \Gamma^A_{BC} (\nabla_a f^B)(\nabla_b f^C) g^{ab} = 0.$$

- $m = 1 \Rightarrow$ harmonic map eqns are precisely geodesic eqns on N
- $n = 1, \mathbf{h}$ Euclidean $\Rightarrow$ harmonic map eqns describe harmonic function on M
- $\mathbf{g}$ Lorentzian $\Rightarrow$ harmonic map = wave map

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How about some connections with General Relativity?

General Relativity: Einstein Equations (1915)

$$R_{\mu\nu} - \frac{R}{2}\mathbf{g}_{\mu\nu} + \Lambda\mathbf{g}_{\mu\nu} = T_{\mu\nu}, \qquad \mu,\nu = 0,\dots 3$$

- $(M, \mathbf{g})$ Lorentzian manifold, $R_{\mu\nu}$ Ricci curvature tensor of $\mathbf{g}$, R Ricci scalar curvature of g (this talk: $\Lambda, T_{\mu\nu} = 0$)
- Vacuum Equations $R_{\mu\nu} = 0$, quasilinear system in $(\mathbf{g}, \partial\mathbf{g}, \partial^2\mathbf{g})$
- Exact solutions? Schwarzschild: static, spherically symmetric AF (1916), Kerr: stationary, axisymmetric AF (1963)
- Symmetry reductions correspond to existence of Killing fields for the metric $\mathbf{g}$ (timelike KF = stationary, rotation $so(3)$= spherical symmetry, rotation $so(2)$ = axisymmetry)

- Two-Killing field reduction gives the Ernst Equation ('68)

$$(\varepsilon + \bar{\varepsilon})\triangle\varepsilon + 2\nabla\varepsilon \cdot \nabla\varepsilon = 0.$$

Key: Ernst equation is derivable from the harmonic map action

$$f : (\mathbb{R}^3, \mathbf{g}) \quad \longrightarrow \quad (\mathbb{H}_{\mathbb{R}}, \mathbf{h})$$
$$ds^2_{\mathbf{g}} = d\rho^2 + dz^2 + \rho^2 d\phi \qquad ds^2_{\mathbf{h}} = (\mathfrak{Re}\,\varepsilon)^{-2}|d\varepsilon|^2.$$

⇒ SAS EVE describe equations for a harmonic map

- (Zakharov-Belinski '78): 2 Killing fields ⇒ metric g block-diagonal

$$g_{\mu\nu}dx^\mu dx^\nu = f(\rho, z)(d\rho^2 \pm dz^2) + \tilde{g}_{ab}(\rho, z)dx^a dx^b \quad a, b = 1, 2$$

Then there exists a linear system U, V for which equations for $\tilde{g}$ appear as compatibility condition

⇒ SAS EVE is Lax-integrable

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A harmonic mapping (equiv., a system of PDEs which can be described as a harmonic mapping) is called completely integrable when the n-dimensional manifold of fields N admits $2n-1$ linearly independent Killing fields.

The idea is that one may be able to use $2n$ integrals of geodesic motion arising from $2n-1$ Killing fields and the metric g to solve the geodesic equations of N completely algebraicially–not just by quadratures. This lies in parallel with what happens in the Lax-integrable case, where ISM often reduces the Gelfand-Levitan-Marchenko integrals to an algebraic system.

complete integrability $\leftrightarrow$ Lax integrability?

Some Examples (as “evidence” for the Conjecture)

- SAS EVE as harmonic map $f : \mathbb{R}^3 \to \mathbb{H}_{\mathbb{R}} \cong SL(2,\mathbb{R})/SO(2)$
  Target is 2-dimensional and admits 3=2(2)-1 independent KFs
  System is indeed Lax-integrable (Z-B, ‘78)
- SAS E-Maxwell as harmonic map $f : \mathbb{R}^3 \to \mathbb{H}_{\mathbb{C}} \cong SU(2,1)/S(...)$
  Target is 4-dimensional and admits 8 $>$ 2(4)-1 independent KFs
  System is Lax-integrable (Alekseev ‘80, Eris-Gürses-Karasu ‘83)
- $SU(n)$ source-free SDYM in 2D as harmonic map $f : \mathbb{R}^2 \to \mathcal{N}$
  Here, $\mathcal{N}$ is n^2-1-dimensional and admits 2(n^2-1)$>$2n-1 KFs
  System is Lax-integrable (Gürses-Jantzen-Xanthopoulos ‘83)
- For $n = 2, 3$, manifold with $2n - 1$ KFs is maximally symmetric.
  Classification of harmonic maps into maximally symmetric spaces gives 3 types. Compact ones are Lax-integrable (Zakharov-Mihailov ‘78, Uhlenbeck ‘03, Uhl-Terng ‘10)

The definition of complete integrability requires harmonic map formulation. Does complete integrability $\Rightarrow$ Lax- integrability encode an intermediate step we can try to understand?

Notice that for (SAS) EVE and EM,

$$\begin{aligned} \mathbb{H}_{\mathbb{R}} &\cong SL(2,\mathbb{R})/SO(2) \cong SU(1,1)/S(U(1)\times U(1)) \\ \mathbb{H}_{\mathbb{C}} &\cong SU(2,1)/S(U(2)\times U(1)). \end{aligned}$$

The Punchline: EVE/EM + 2 Killing fields Lax-integrable because Harmonic Maps into (certain) symmetric spaces are Lax-integrable.

Theorem 1 (SB, S. Tahvildar-Zadeh 2012) Let G be a real semisimple Lie group and K a maximal compact subgroup. Then any axially symmetric harmonic map from $\mathbb{R}^3$ into the Riemannian symmetric space G/K satisfies a Lax integrable system of equations.

Theorem 2 (SB, S. Tahvildar-Zadeh, 2013) If the Lie group G is such that the two involutions τ and σ can be given by conjugation with the same element, then the vesture method can be used to construct new harmonic maps with any number of prescribed singularities starting from any given harmonic map.

The two Theorems, in a diagram

$$
\begin{array}{ccc}
q_0(\vec{x}) & \xrightarrow[\Psi|_{\lambda=0}=q_0]{D\Psi=\Omega\Psi} & \Psi_0(\vec{x},\lambda) \\
\downarrow & & \Big\downarrow {\scriptstyle \Psi=\chi\Psi_0} \\
q(\vec{x}) & \xleftarrow[\lambda=0]{} & \Psi(\vec{x},\lambda)
\end{array}
$$

arXiv:[1312.5253], [1209.1383], Nonlin. Sys. Compl, v10, XVI (2014)

Obstructions and Next Steps (Geometric Analysis)

- Xanthopoulos' notion of complete integrability "doesn't see" the dimensionality of manifold of coordinates M
- Current Lax mechanism applies when the base manifold is *effectively* two-dimensional and the manifold of fields is a Riemannian symmetric space

◇ Connect the number of KFs to symmetric space formulation concretely, starting with maximally symmetric spaces

Obstructions and Next Steps (General Relativity)

- Key will be to understand “admissible singular structures” for the Lax-integrable gravitational examples. Broader class of examples may help distill geometric features (KF) from integrable ones (RS bundle)

◇ 2-soliton EVE currently under investigation

◇ Kaluza-Klein reductions of SUGRA will require lifting of technical restrictions on involutions σ, τ

Table : Nonlinear σ-Models $f : \mathcal{M} \longrightarrow G/K$ [Breit. et. al. '88]

Gravitational Theory	G/K	n-Soliton Dressing
Einstein-Maxwell $m = 3+1$, $k = 2$ ($D = 2$ SUGRA)	$\frac{SU(2,1)}{S(U(2)\times U(1))}$	Kerr-Newman
Einstein gravity $m = n+4$, $k = n$	$\frac{SL(n+2)}{SO(n+2)}$	$\frac{SU(p,q)}{S(U(p)\times U(q))}$ [SB-STZ]
$m = 3+1$, $k = 2$ ($D = 4$ SUGRA)	$\frac{SO(8,2)}{SO(8)\times SO(2)}$	?
$m = 5$, $k = 2$ or 3 ($D = 1$ SUGRA)	$\frac{G_{2(2)}}{SL(2,\mathbb{R})\times SL(2,\mathbb{R})}$	[Figueras, et.al. '10]
$m = 3+1$, $k = 2$ ($D = 8$ SUGRA)	$E_{8(+8)}/SO^*(16)$	$E_{7(+7)}/SU(8)$?

Thank you very much!

RUTGERS

and

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