The Non-Lefschetz Locus

Special session 12 on Commutative Artinian Algebras and Their Deformations

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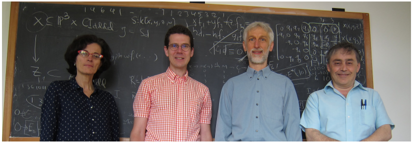

Lefschetz Properties

Theorem (Hard Lefschetz Theorem)

$$H^{n-k}(X) \xrightarrow[\cong]{L^k} H^{n+k}(X)$$

when X is a smooth variety in $\mathbb{P}^r_{\mathbb{C}}$ and $n = \dim X$.

Definition

A standard graded artinian algebra A has the Weak Lefschetz Property (WLP) if for some $\ell \in [A]_1$

$$\times\ell\colon [A]_i \longrightarrow [A]_{i+1}$$

has maximal rank for all i. It has the Strong Lefschetz Property (SLP) if

$$\times\ell^k\colon [A]_i \longrightarrow [A]_{i+k}$$

has maximal rank for all i and k.

General and special complete intersections

It is known that

- all monomial complete intersections have the WLP and SLP (Stanley)
- and it follows that the general complete intersection has it
- all complete intersections in $k[x, y, z]$ have the WLP. (Harima, Migliore, Nagel and Watanabe)

Remark

The WLP does not distinguish between the general complete intersection and the monomial complete intersection.

It is unknown whether

- all complete intersections in $k[x, y, z, w]$ have the WLP.
- all Gorenstein algebras in $k[x, y, z]$ have the WLP.

The non-Lefschetz locus

Definition

Given an artinian algebra $A = k[x_1, x_2, \ldots, x_n]/I$, the non-Lefschetz locus, $\mathscr{L}_{I,i}$ is the set of linear forms ℓ in $(\mathbb{P}^{n-1})^*$ such that

$$\times\ell\colon [A]_i \longrightarrow [A]_{i+1}$$

fails to have maximal rank.

Remark

The non-Lefschetz locus is cut out by determinantal conditions which makes it into a subscheme of $(\mathbb{P}^{n-1})^*$.

Proposition

If $h_A(i) \leq h_A(i+1) \leq h_A(i+2)$ and $[\operatorname{Soc} A]_i = 0$ then $I(\mathscr{L}_{I,i+1}) \subseteq I(\mathscr{L}_{I,i})$.

Expected codimension and degree

Since $I(\mathscr{L}_{l,i})$ is determinantal we have

$$\operatorname{codim} \mathscr{L}_{l,i} \le \min\{h_{i+1} - h_i + 1, n\}$$

if $h_i \le h_{i+1}$. We say that $\mathscr{L}_{l,i}$ has the expected codimension if we have equality and then the degree is known to be $\binom{h_{i+1}}{h_{i+1}-h_i+1}$.

Remark

For Gorenstein algebras, only the middle degrees are relevant.

Conjecture

For a non-empty open subset of the space of complete intersections of type $(d_1, d_2, \ldots, d_n)$, $\mathscr{L}_l$ has the expected codimension.

Monomial complete intersections

Theorem

Let $I = \langle x_1^{d_1}, x_2^{d_2}, \ldots, x_n^{d_n} \rangle \subseteq k[x_1, x_2, \ldots, x_n]$ with $2 \leq d_1 \leq d_2 \leq \cdots \leq d_n$. Then the following characterization holds:

1. *If $d_n \geq \frac{\sum d_i - n + 1}{2}$, then $\ell = \sum a_i x_i$ is in $\mathscr{L}_I$ if and only if $a_n = 0$.*
2. *If $d_n \leq \frac{\sum d_i - n + 1}{2}$ and the socle degree is even, then $\ell = \sum a_i x_i$ is in $\mathscr{L}_I$ if and only if $a_j = 0$ for some j with $d_j > 2$ or $a_i = 0$ for at least two indices j with $d_j = 2$.*
3. *If $d_n \leq \frac{\sum d_i - n + 1}{2}$ and the socle degree is odd then $\ell = \sum a_i x_i$ is in $\mathscr{L}_I$ if and only if $a_i = 0$ for some index i.*

Remark

The codimension can be one or two and the non-Lefschetz locus does not need to be unmixed.

General complete intersections

Theorem

The non-Lefschetz locus $\mathscr{L}_I$ has the expected codimension for $I = \langle F_1, F_2, F_3, F_4 \rangle \subseteq k[x_1, x_2, x_3, x_4]$ in a non-empty open subset of the space of complete intersections of forms of degrees $d_1 \leq d_2 \leq d_3 \leq d_4$.

Remark

In particular, the non-Lefschetz locus is non-empty if and only if $h_{\lfloor \frac{e+1}{2} \rfloor} - h_{\lfloor \frac{e-1}{2} \rfloor} \leq 2$, where $e = \sum_{d_i} -4$ is the socle degree. This happens when $d_4 \geq d_1 + d_2 + d_3$ or $-d_1 + d_2 + d_3 \leq d_4 \leq d_1 + d_2 + d_3$ and $d_1 + d_2 + d_3 - d_4 \leq 4$ or $d_4 \leq -d_1 + d_2 + d_3$ and $d_1 \leq 2$.

Proof.

We use an incidence correspondence, linkage and a dimension formula by Conca and Valla. □

General Gorenstein algebras in three variables

Theorem

Let $A = k[x,y,z]/I$ be a "general" artinian Gorenstein algebra with h-vector $h = (h_0, h_1, \dots, h_e)$.

1. *If $h_i = h_{i+1}$ for some $1 \leq i < e$ then $\mathscr{L}_I$ has codimension one, which is the expected codimension.*
2. *If e is even and $h_{e/2} > h_{e/2-1}$ then $\mathscr{L}_I$ has the expected codimension if and only if*

$$(g_0, g_1, \dots, g_{e/2}) = (h_0, h_1 - h_0, \dots, h_{e/2} - h_{e/2-1})$$

   is an h-vector of decreasing type.

Proof.

We use points in UPP and the structure theorem by Buchsbaum and Eisenbud which forces a GCD when we don't have decreasing type. □