Symmetric decomposition of the associated graded algebra of a Gorenstein Artinian algebra

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Joint work with Anthony Iarrobino

Gorenstein Artinian algebras

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Let A be an Artinian algebra, obtained as a quotient of a polynomial ring $R = k[x_1, \ldots, x_r]$ by an ideal I (not necessarily homogenous).

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The *socle degree* of a Gorenstein Artinian (GA) algebra A is the highest integer j such that

$$\mathrm{m}^j \neq 0.$$

Gorenstein Artinian algebras

Associated graded algebra

The graded associated algebra of a GA algebra A is

$$A^* := \bigoplus_{i \geq 0} \frac{\mathfrak{m}^i}{\mathfrak{m}^{i+1}}.$$

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whose successive quotients $Q(a) = C(a)/C(a+1)$ are reflexive A^*-modules.

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If A is graded, its Hilbert function is symmetric, e.g.

$$A = k[x, y, z]/(z^2, y^2, x^3 - xyz), \quad H(A) = (1, 3, 4, 3, 1)$$

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but admits a symmetric decomposition

$H(A)$	1	3	3	1	1
$H\big(Q_A(0)\big)$	1	1	1	1	1
$H\big(Q_A(1)\big)$	0	2	2	0	

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Iarrobino 1994 There is an exact pairing

$$Q_A(a)_i \times Q_A(a)_{j-a-i} \longrightarrow k$$

Gorenstein Artinian algebras

A few questions

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3. Can $Q_A(a)$ be an acyclic module? Is it possible to have

$$H\big(Q_A(a)\big) = (0, s, 0, \ldots, 0, s, 0)?$$

Dual ring

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Denote by $\mathfrak{D} = k_{DP}[X_1, \ldots, X_r]$ the divided power ring and let the polynomial ring $R = k[x_1, \ldots, x_r]$ act on $\mathfrak{D}$ by contraction:

$$x_i^{\alpha} \circ X_i^{[\beta]} = \begin{cases} X_i^{[\beta-\alpha]} & \text{if } \beta \geq \alpha, \\ 0 & \text{if } \beta < \alpha. \end{cases}$$

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Macaulay 1916 Giving an ideal I of the polynomial ring R defining an Artinian quotient $A = R/I$ of length $\dim_k(A) = n$ is equivalent to giving a length-n R-submodule A' of the divided power algebra $\mathfrak{D}$.

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$$f \in \mathfrak{D} \mapsto R/\operatorname{Ann} f.$$

Gorenstein sequences

Typical Hilbert functions

If f is a general polynomial of degree j, then the Hilbert function of $A = R/\operatorname{Ann} f$ is maximal:

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$j = 2$

1	1	1
1	2	1
1	3	1
1	4	1
1	5	1

$j = 3$

1	1	1	1
1	2	2	1
1	3	3	1
1	4	4	1
1	5	5	1

$j = 4$

1	1	1	1	1
1	2	3	2	1
1	3	6	3	1
1	4	10	4	1
1	5	15	5	1

$j = 6$

1	1	1	1	1	1	1
1	2	3	4	3	2	1
1	3	6	10	6	3	1
1	4	10	20	10	4	1
1	5	15	35	15	5	1

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In general,

$$1 \quad r \quad \binom{r+1}{2} \quad \binom{r+2}{3} \quad \cdots \quad \binom{r+2}{3} \quad \binom{r+1}{2} \quad r \quad 1$$

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$\dim \mathfrak{D}_i$ $\qquad$ $\dim R_i$

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Easy atypical Hilbert functions

Connected sums

$$f = X^{[3]}Y^{[3]}, \quad g = Z^{[5]} + W^{[5]}$$

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Partials of f

			$X^{[3]}$			
		$X^{[2]}$	$X^{[2]}Y$	$X^{[3]}Y$		
	X	XY	$XY^{[2]}$	$X^{[2]}Y^{[2]}$	$X^{[3]}Y^{[2]}$	
1	Y	$Y^{[2]}$	$Y^{[3]}$	$XY^{[3]}$	$X^{[2]}Y^{[3]}$	f

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Partials of g

	Z	$Z^{[2]}$	$Z^{[3]}$	$Z^{[4]}$	
1	W	$W^{[2]}$	$W^{[3]}$	$W^{[4]}$	g

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Gorenstein sequences

Ubiquity

Question.

If we have a given Gorenstein Artinian algebra A, is there another Gorenstein Artinian algebra B such that

$$H\big(Q_B(u)\big) = \begin{cases} H\big(Q_A(u)\big), & \text{for } u < a, \\ 0, & \text{for } u \geq a, \end{cases}$$

for a given integer a?

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$H(B)$	1	•	•	•	•	•	1
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Stronger question.

If we have a Gorenstein Artinian algebra $A = R/I$, is $R/\mathcal{C}(a)$ still a Gorenstein Artinian algebra?

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- Yes, to both, if codim $R/\mathcal{C}(a) \leq 2$.

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- Yes, to both, if codim $R/\mathcal{C}(a) \leq 2$.
- No, to both, if codim $R/\mathcal{C}(a) \geq 3$.

Question.
Is it possible to have

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Iarrobino 1994 There is a complete intersection A with

$H(A)$	1	3	3	4	2	1	1
$Q(0)$	1	1	1	1	1	1	1
$Q(1)$	0	1	2	2	1	0	
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Can we create other such examples?

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$H(R/\operatorname{Ann} f)$	1	2	3	4	5	4	3	2	1

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$H(R/\operatorname{Ann} g)$	1	4	4	4	4	4	4	1	
$H\big(R/\operatorname{Ann}(f+g)\big)$	1	4	5	4	5	6	3	2	1

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$H(R/\operatorname{Ann} g)$	1	4	4	4	4	4	4	1	
$H\big(R/\operatorname{Ann}(f+g)\big)$	1	4	5	4	5	6	3	2	1
$Q(0)$	1	2	3	4	5	4	3	2	1
$Q(1)$	0	0	0	0	0	0	0	0	
$Q(2)$	0	2	0	0	0	2	0		
$Q(3)$	0	0	0	0	0	0			
$Q(4)$	0	0	2	0	0				

Theorem (Iarrobino, __)

Let $f, h_1, \dots, h_s \in k_{DP}[X_1, \dots, X_r]$ *be homogeneous polynomials with*

- $\deg f = j$,
- $\deg h_t = k_t$,
- $j - 2 \geq k_1 \geq \cdots \geq k_s \geq 1$.

Let $a_t = j - (k_t + 1)$ *and consider the polynomial*

$$F = f + h_1 Z_1 + \cdots + h_s Z_s.$$

Then symmetric decomposition of the GA algebra $A = R/\operatorname{Ann} F$ *satisfies*

$$Q(u) = 0 \text{ for } u \notin \{0, a_1, \dots, a_s\} \cup \{a_{t_1} + a_{t_2} \mid 1 \leq t_1 \leq t_2 \leq s\}$$

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Gorenstein sequences

Non-ubiquity

$$f = X^{[3]}Y^{[3]}, \quad g = X^{[4]}Z_1 + Y^{[4]}Z_2$$

$H\big(R/\operatorname{Ann}(f+g)\big)$	1	4	5	4	5	2	1
$Q(0)$	1	2	3	4	3	2	1
$Q(1)$	0	2	0	0	2	0	
$Q(2)$	0	0	2	0	0		
$H\big(R/\mathcal{C}(2)\big)$	1	4	3	4	5	2	1

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Proposition (Iarrobino, __)

The sequence $H = (1, 4, 3, 4, 5, 2, 1)$ does not occur as the Hilbert function of a GA algebra.

Museu de Arte Contemporânea de Serralves, Álvaro Siza Vieira, 1997

Fotografia: Nelson Alexandre Rocha, *Serralves minimalista #5*, 2007