Symmetries and defects in 3d TFT

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Reminder:

1. TFT is a symmetric monoidal functor $\quad tft : cob_{n,n-1} \longrightarrow \mathcal{S}$

$n=2 \quad tft(S^1)$ is a commutative Frobenius algebra

Here: extended 3d TFT is a symmetric monoidal 2-functor

$$tft : cob_{3,2,1} \longrightarrow 2\text{-}vect$$ (see next slide)

2. 2d TFT with boundaries / defects leads to modules / bimodules and thus relates to representation theory.

$cob_{2,1}$ enlarged to $cob^{\partial}_{2,1}$ (see next slide)

Topic of this talk: 3d extended TFT relates to categorified representation theory

3d extended TFT

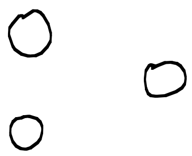
Generalization of definition:

$$\mathrm{tft}: \quad \mathrm{cobord}_{3,2,1} \longrightarrow \text{2-vect}(\mathbb{C})$$

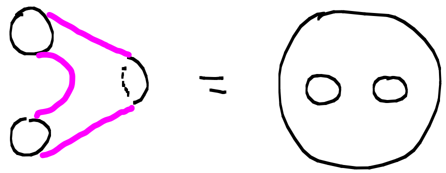
symmetric monoidal bifunctor

- $\text{2-vect}(\mathbb{C})$: finitely semi-simple, $\mathbb{C}$-linear abelian categories
- $\mathrm{cobord}_{3,2,1}$

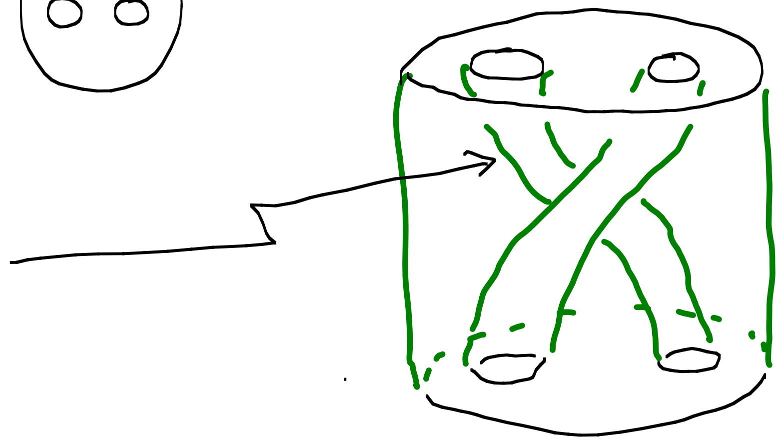
objects



1-morphisms



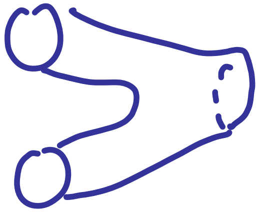
2-morphisms:
mfds. with corner

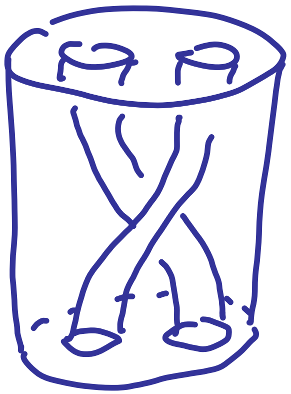


Wilson lines /
ribbon graphs

Evaluation of the bifunctor tft:

- $\mathrm{tft}\Big($  $\Big) =: \mathcal{C}$ — a finitely ssi $\mathbb{C}$-linear category

- $\mathrm{tft}\Big($  $\Big)$ — functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

- $\mathrm{tft}\Big($  $\Big)$ — natural transformation

$$\begin{tikzcd} \mathcal{C} \times \mathcal{C} \arrow[r, bend left=50, "\otimes"] \arrow[r, bend right=50, "\otimes^{opp}"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{C} \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{tft} \arrow[r, dashed, bend left, "\text{Freed, ...}"] & \mathcal{C} = \mathrm{tft}(S^1) \arrow[l, bend left, "\text{RT}"] \end{tikzcd}$$

modular tensor category (MTC)

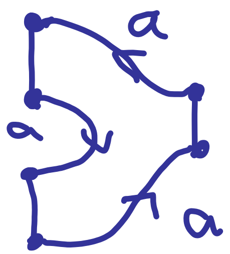
2 d TFT with defects

2d TFT with boundaries / defects leads to modules / bimodules and thus relates to representation theory.

$\mathrm{cob}_{2,1}$ enlarged to $\mathrm{cob}^{\partial}_{2,1}$

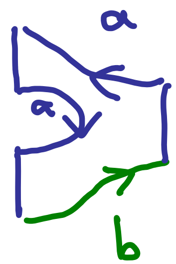
For $d = 2$

$\mathrm{tft}\left(I^{a}_{a} \right) =: A_a$



$\uparrow$

algebra



$\mathrm{tft}\left(I^{a}_{b} \right)$ is A_a- module

Center $Z(A_a) = \mathrm{tft}(S^1) = C$

Boundary conditions $= A_a$-mod

Category of A_a- modules

$\mathrm{tft}\left(I^{b_2}_{b_1} \right) = \mathrm{Hom}_{A_a\text{-mod}}(b_1, b_2)$

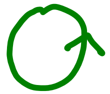
Topic of this talk: 3d extended TFT relates to categorified representation theory:

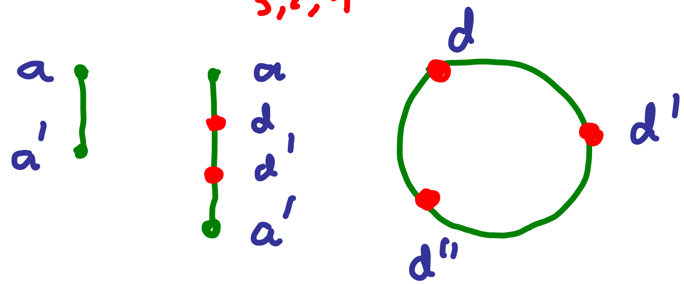
module categories and bimodule categories over fusion categories

Extended 3d TFTs with boundaries and defects

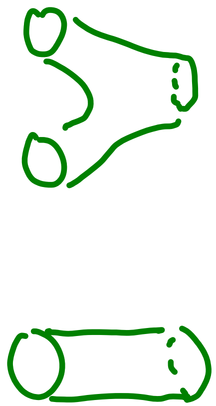
$\mathrm{Cobord}_{3,2,1}$ $\mathrm{Cobord}^{\partial}_{3,2,1}$

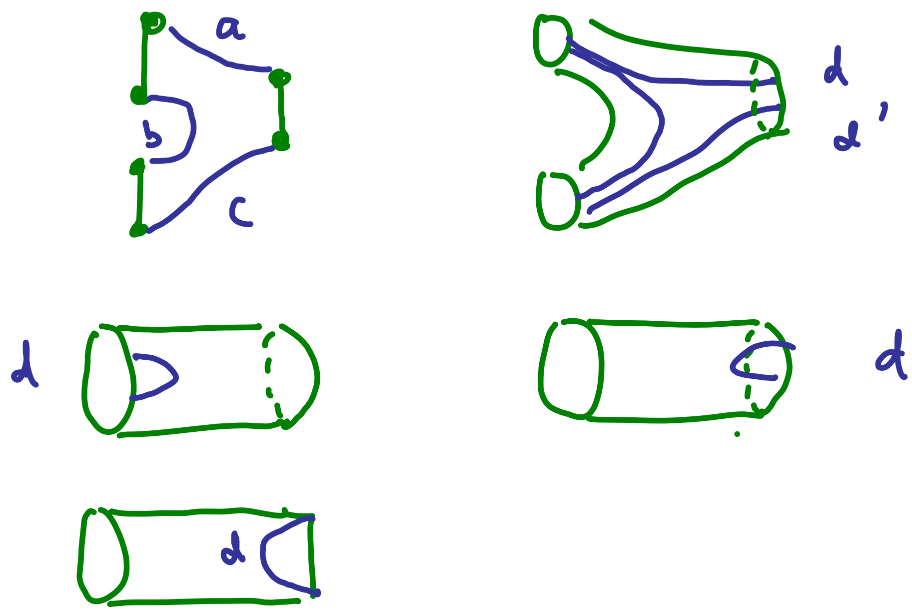
Objects





1-Morphisms





2-Morphisms

many

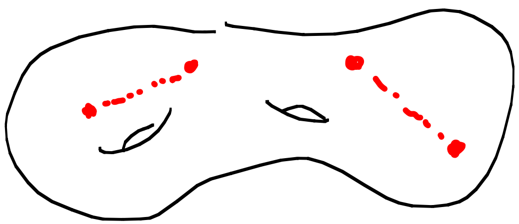
many more

Application 1: Quantum codes from twist defects

- Σ surface $\Rightarrow$ $\mathcal{H}_G(\Sigma)$ quantum code ($G = \mathbb{Z}_2$: toric code)
- Representation of braid group gives quantum gates

Problems:
- low genus of Σ $\rightarrow$ small codes
- simple systems $\rightarrow$ no universal gates

Idea: Bilayer systems $\mathcal{C} \boxtimes \mathcal{C}$ and twist defects create branch cuts



(cf. permutation orbifolds)

Application of defects: relative field theories

A relative field theory is a (d-1)-dimensional theory on the boundary or on a codimension one defect in a d-dimensional theory.

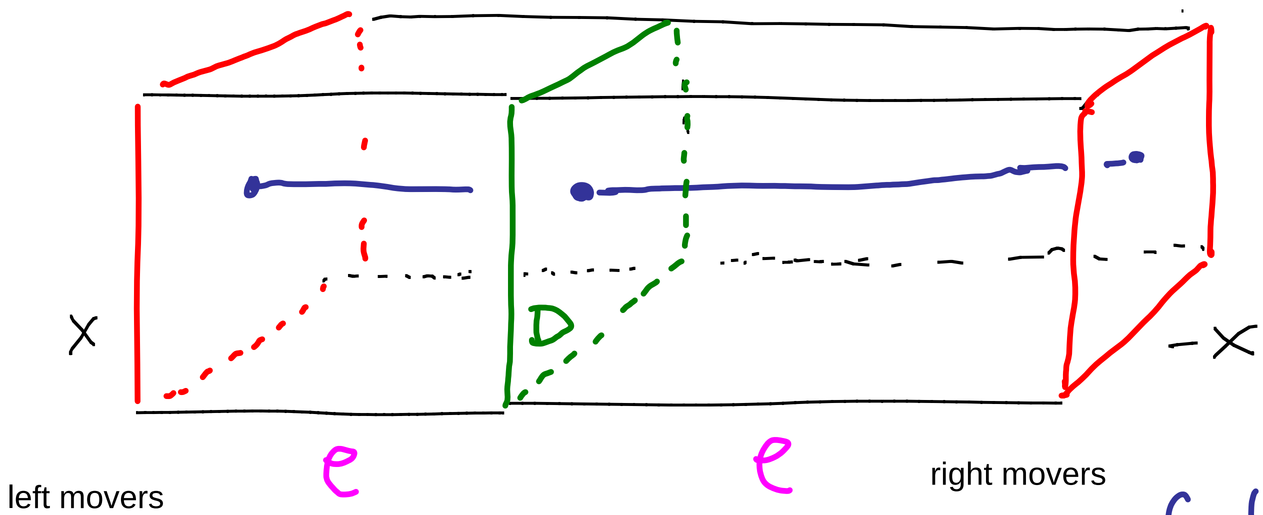
Important case: d-dimensional theory is a topological field theory

Situation 1:

d-dimensional theory is "invertible" TFT $\rightsquigarrow$ anomalous theory in (d-1) dimensions

Situation 2:

d=3, TFT of Reshetikhin-Turaev type $\rightsquigarrow$ TFT construction of 2d RCFT correlators



Category $\mathcal{C}$ is representation category of chiral symmetries for RCFT: a modular tensor category

$$\mathrm{Cor}(X) \in \mathrm{tft}_{\mathcal{C}}(X) \otimes \mathrm{tft}_{\mathcal{C}}(-X)$$

1. TFT of Turaev-Viro type

Input: (spherical) fusion category $\mathcal{A}$

3d TFT, e.g. by state sum construction

$$\mathrm{tft}_{\mathcal{A}}(S^1) = Z(\mathcal{A})$$

Drinfeld center of $\mathcal{A}$

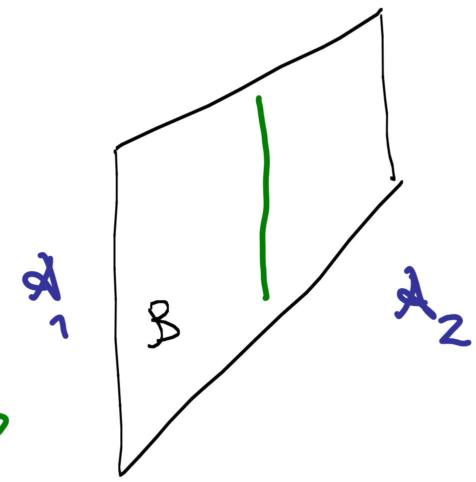
$$= \{ (U, c_X : U \otimes X \xrightarrow{\sim} X \otimes U) \}$$

Example: G finite group, $\mathcal{A} = G\text{-vect}$ (G graded f.d. $\mathbb{C}$-vector spaces)

$$\mathrm{tft}_G : \mathrm{Cob}_{3,2,1} \xrightarrow{\mathrm{Bun}_G} \mathrm{SpanGrp} \xrightarrow{\text{linearize}} 2\text{-vect}$$

Analysis [FSV, ...]

$\mathcal{A}_1$-$\mathcal{A}_2$ defect labelled by $\mathcal{A}_1$-$\mathcal{A}_2$ bimodule categories



Labels for Wilson lines: $\mathrm{End}_{\mathcal{A}_1,\mathcal{A}_2}(\mathcal{B})$

Defects and boundaries in Dijkgraaf-Witten theories

Idea: keep the same 2-step procedure, but allow for more general "bundles" as field configurations

$$\mathrm{tft}_G : \mathrm{Cob}^{\partial}_{3,2,1} \xrightarrow{\mathrm{Bun}_G} \mathrm{SpanGrp} \xrightarrow{\text{linearize}} 2\text{-}\mathrm{Vect}$$

$\mathrm{Cob}^{\partial}_{3,2,1}$: mfd w/ cell decomposition

Idea: (generalizations of) relative bundles [FPSV `03]

Given relative manifold $j : Y \to X$ and group homomorphism $i : H \to G$

$$\mathrm{Bun}_{H\to G}(Y \to X) = \left\{ \begin{array}{c} P_G \\ \downarrow \\ X \end{array} , \begin{array}{c} P_H \\ \downarrow \\ Y \end{array} , \ \alpha : \mathrm{Ind}_H^G P_H \xrightarrow{\sim} j^* P_G \right\}$$

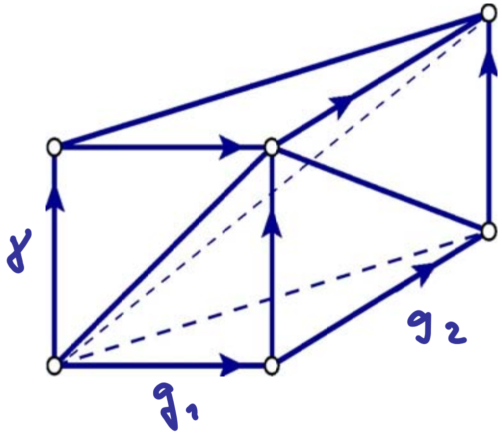
Additional datum: twisted linearization

$$\omega \in Z^3(G, \mathbb{C}^*) \cong Z^3(*//G, \mathbb{C}^*)$$

Transgress to loop groupoid $[*//\mathbb{Z}, *//G] \cong G//G$

$$\tau(\omega) \in Z^2(G//G, \mathbb{C}^*)$$

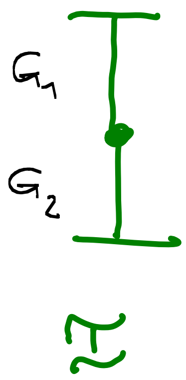
twisted linearization



$\tau(\omega)(\gamma; g_1, g_2)$

Module categories from DW theories

Example: Interval



$i_1 : H_1 \to G_1$

$H_{12} \to G_1 \times G_2$

$i_2 : H_2 \to G_2$

$$\mathrm{Bun}(\mathcal{I}) \cong {}_{G_1 \times G_2}\backslash\backslash^{\,G_1 \times G_1 \times G_2 \times G_2} // {H_1 \times H_{12} \times H_2}$$

Transgress to 2-cocycle on $\mathrm{Bun}(\mathcal{I})$

(for twisted linearization)

Data:

$\omega_a \in Z^3(G_a, \mathbb{C}^*)$ bulk Lagrangian

$\theta_a \in C^2(H_a, \mathbb{C}^*)$ bdry Lagrangian

$d\theta_a = i^* \omega_a$

$\theta_{12} \in C^2(H_{12}, \mathbb{C}^*)$

$d\theta_{12} = \omega_1 (\omega_2)^{-1}$

Check:

Module category $\mathcal{M}(H, \theta)$ over $(G\text{-vect})^{\omega}$

$$\mathcal{H}\left(\mathcal{I}_{H_1}^{H_2}{}_{G} \right) = \mathrm{Fun}_{(G\text{-vect})^{\omega}}\left(\mathcal{M}(H_1, \theta_1), \mathcal{M}(H_2, \theta_2) \right)$$

This "explains" a representation theoretic result: classification of module categories, cf. [Ostrik]

3. A construction in representation theory [ENOM]

$\mathcal{A}$ fusion category, $\mathcal{B}$ a $\mathcal{A}$-bimodule category

Given $(a,c) \in Z(\mathcal{A})$, $\mathbb{C}$-linear functors

$$R_a: \mathcal{B} \longrightarrow \mathcal{B} \qquad\qquad L_a: \mathcal{B} \longrightarrow \mathcal{B}$$
$$b \longmapsto b \otimes a \qquad\qquad b \longmapsto a \otimes b$$

Half-braiding c => R_a, L_a are module functors. We thus get:

$$R: Z(\mathcal{A}) \longrightarrow \mathrm{End}_{\mathcal{A}\mathcal{A}}(\mathcal{B}) \qquad\qquad L: Z(\mathcal{A}) \longrightarrow \mathrm{End}_{\mathcal{A}\mathcal{A}}(\mathcal{B})$$
$$a \longmapsto R_a \qquad\qquad a \longmapsto L_a$$

monoidal functors

Fact: $\mathcal{B}$ invertible $\quad \left(\mathcal{B} \boxtimes_{\mathcal{A}} \tilde{\mathcal{B}} \simeq \mathcal{A} \text{ for some } \tilde{\mathcal{B}} \right)$

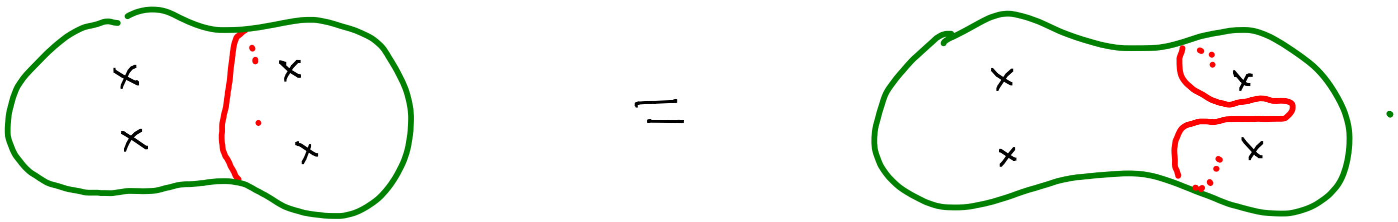
then $\quad Z(\mathcal{A}) \xrightarrow[R]{\sim} \mathrm{End}(\mathcal{B}) \xleftarrow[L]{\sim} Z(\mathcal{A})$

Thus: $\quad \mathrm{BrPic}(\mathcal{A}) \xrightarrow[\text{ENOM}]{\sim} \mathrm{BraidEq}(Z(\mathcal{A}))$

Understand and use this in 3d TFT with defects

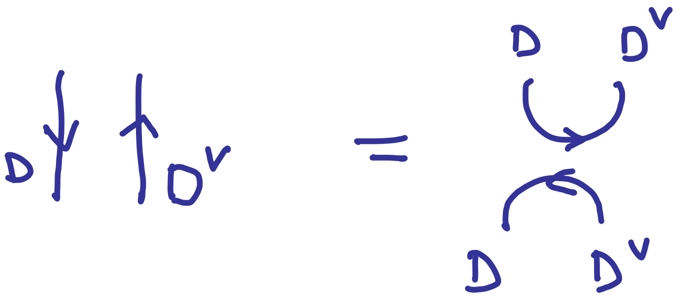
4. General remark: invertible topological defects and symmetries

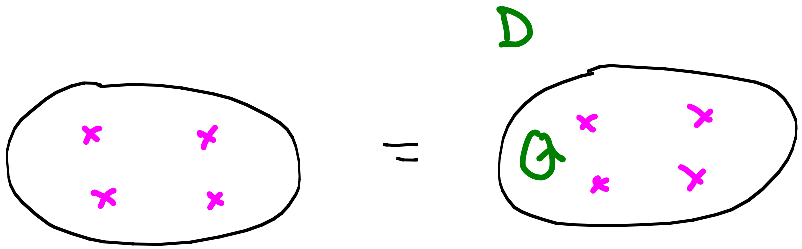
Topological defects: Correlators do not change under small deformations of the defect



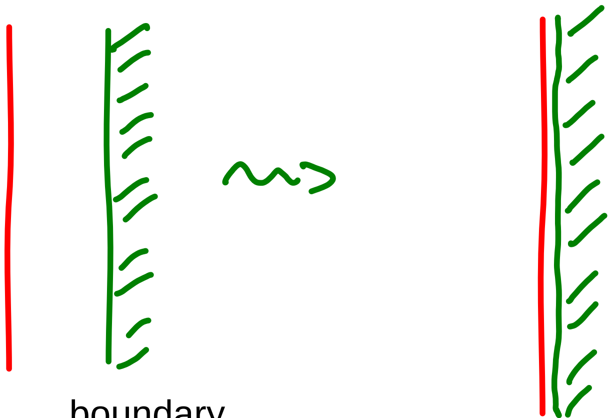
Symmetries from invertible topological defects (2d RCFT [FFRS '04])

invertible defects



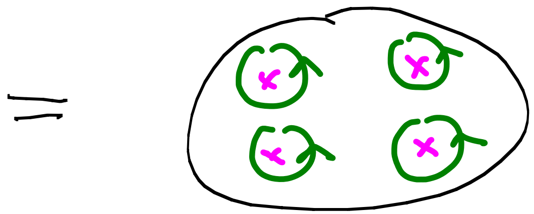


Action on boundaries:



defect boundary composite boundary

equality of correlators



Insight:
group of invertible topological line defects acts as a symmetry group in two-dimensional theories

5. Symmetries in 3d extended TFTs from defects

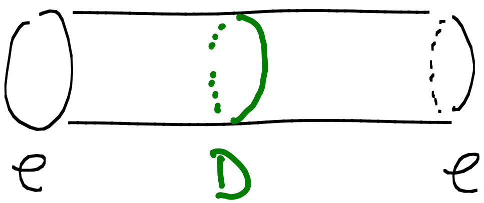
Symmetries $\leftrightarrow$ invertible topological defects

For 3d TFT:

Symmetries for $tft_{\mathcal{C}}$ with $\mathcal{C} = \mathcal{Z}(\mathcal{A})$ are invertible $\mathcal{A}$-bimodule categories

Bicategory ("categorical 2-group") $BrPic(\mathcal{A})$, the Brauer-Picard group

Important tool: Transmission functor



Braided equivalence, if D invertible

$$BrPic(\mathcal{A}) \underset{[ENO]}{\xrightarrow{\sim}} BrEq(\mathcal{Z}(\mathcal{A}))$$

"Symmetries can be detected
from action on bulk Wilson lines"

Explicitly computable for DW theories:

$$\mathcal{C} = \mathcal{D}(G)\text{-mod} = \mathcal{Z}(G\text{-vect})$$

$\mathcal{D}$ described by

$$H \leq G \times G$$

Linearize span of action groupoids

$$\begin{tikzcd} & H/\!/H \arrow[dl] \arrow[dr] & \\ G/\!/G & & G/\!/G \end{tikzcd}$$

6. A case study: symmetries for abelian DW-theories theories

Special case: : $G = A$ abelian , $\omega = 1$

$$\mathrm{BrPic}(A\text{-vect}) = O_q(A \oplus A^*)$$

with $q(g, \chi) = \chi(g)$

quadratic form

Obvious symmetries:

1) Symmetries of Bun_A

$$\varphi \in \mathrm{Aut}(\mathrm{Bun}_A) = \mathrm{Aut}(A)$$

Subgroup:

$$H_\varphi = \mathrm{graph}\,\varphi \subset A \oplus A, \quad \Theta = 1$$

Braided equivalence:

$$\varphi \oplus (\varphi^*)^{-1} : A \oplus A^* \longrightarrow A \oplus A^*$$

2) Automorphisms of CS 2-gerbe

1-gerbe on Bun_A "B-field"

$$H^2(A, \mathbb{C}^*) \xrightarrow{\sim} AB(A, \mathbb{C}^*) \ni \beta$$

(transgression)

Subgroup: $A_{diag} \subset A \oplus A \qquad \Theta = \beta$

Braided equivalence:

$$A \oplus A^* \longrightarrow A \oplus A^*$$
$$(g, \chi) \longmapsto (g, \chi + \beta(g, -))$$

3) Partial e-m dualities:

Example: A cyclic, fix $\delta : A \xrightarrow{\sim} A^*$

Braided equivalence:

$$A \oplus A^* \longrightarrow A \oplus A^*$$
$$(g, \chi) \mapsto (\delta^{-1}\chi, \delta g)$$

Subgroup: $A_{diag} \subset A \oplus A$

$$\beta(a_1, a_2) = \frac{\delta(a_1)(a_2)}{\delta(a_2)(a_1)} \in AB(A, \mathbb{C}^*)$$

<u>Theorem</u> [FPSV]

These symmetries form a set of generators for $BrPic(A\text{-vect})$.

7. Conclusions

Topological defects are important structures in quantum field theories

- Construction of relative field theories on defects and boundaries

- Applications to topological phases of matter

- Relation to (categorified) representation theory:
  (bi)module categories over monoidal categories

- Defects describe symmetries and dualities

- Symmetry groups as Brauer-Picard groups