INVARIANTS FOR MAPPING CLASS GROUP ACTIONS FROM RIBBON HOPF ALGEBRA AUTOMORPHISMS

joint with J. Fuchs and C. Stigner

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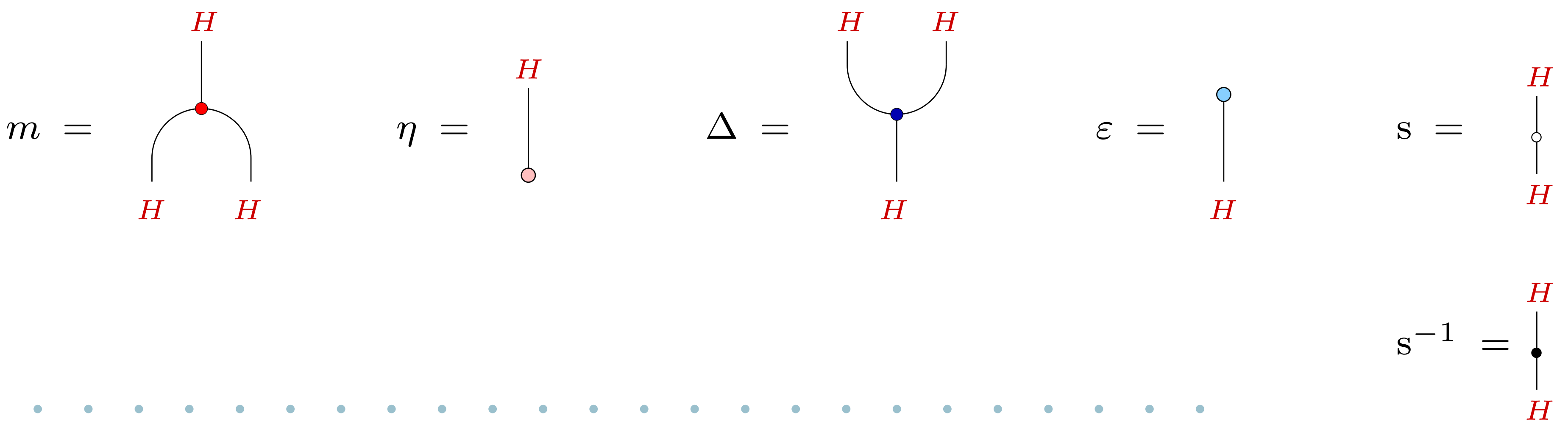
Reminder I : Hopf algebras and graphical calculus

- Hopf algebra $H \equiv (H, m, \eta, \Delta, \varepsilon, \mathrm{s})$ over the field $\mathbb{C}$
- Quasitriangular Hopf algebra (H, R) : R-matrix $R \in H \otimes H$
- Monodromy matrix $Q := R_{21} \cdot R \in H \otimes H$
- Factorizable quasitriangular Hopf algebra :

  $Q = \sum_\ell h_\ell \otimes k_\ell$ with $\{h_\ell\}$ and $\{k_\ell\}$ two vector space bases of H
- This talk: finite-dimensional factorizable ribbon Hopf algebras
- Graphical calculus in the tensor category $\mathcal{V}ect_{\mathbb{C}}$



Reminder II : Coends

- Dinatural transformation

  Given a functor $F\colon\ \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{E}$ and an object $B \in \mathcal{E}$, a dinatural transformation

  $F \Rightarrow B$ is a family of morphisms $\varphi_X\colon\ F(X,X) \to B$ indexed by objects $X \in \mathcal{D}$

s.t.

$$\begin{array}{ccc} F(Y,X) & \xrightarrow{F(\mathrm{id}_Y, f)} & F(Y,Y) \\ {\scriptstyle F(f,\mathrm{id}_X)}\downarrow & & \downarrow{\scriptstyle \varphi_Y} \\ F(X,X) & \xrightarrow{\ \varphi_X\ } & B \end{array}$$

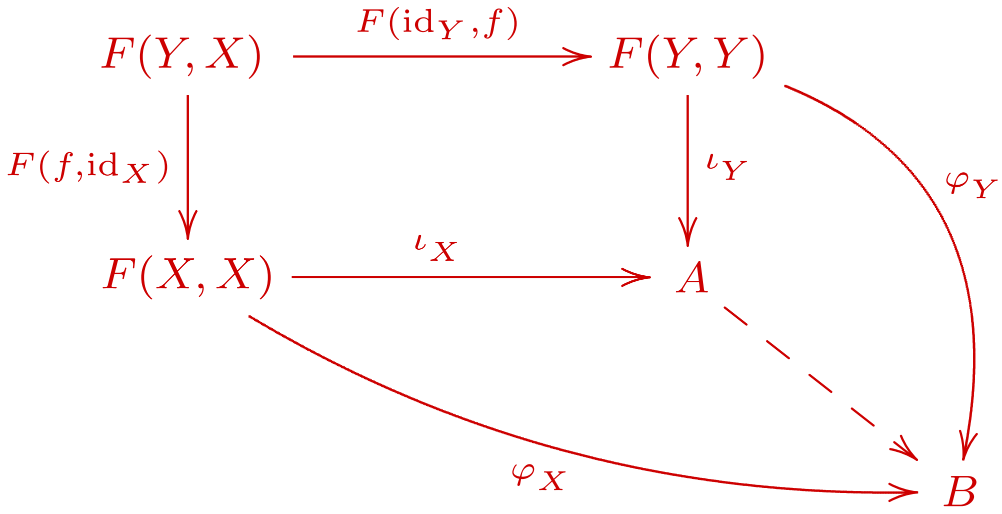
commutes for all morphisms $f\colon X \to Y$

- *Coend* (A, ι) for F:

  initial object in category

  of dinatural transformations

  $F \Rightarrow -$:



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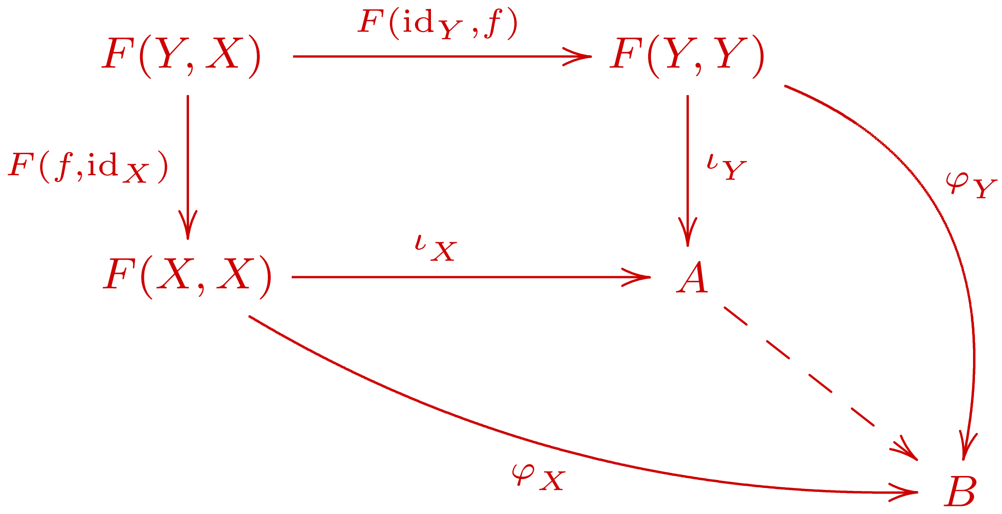

- Notation : $(A,\iota) = \int^X F(X,X)$
- Unique up to unique isomorphism (if exists)
- Coend implements the physical idea of a “sum over all states”

Reminder III : Coends from factorizable ribbon Hopf algebras

- Theorem : $\mathcal{D} = \mathcal{C} \equiv H$-mod$_{fd}$, more generally $\mathcal{D}$ finite abelian $\mathbb{k}$-linear ribbon

category, then the coend $\boxed{\mathcal{H} = \int^U U^\vee \otimes U}$ exists

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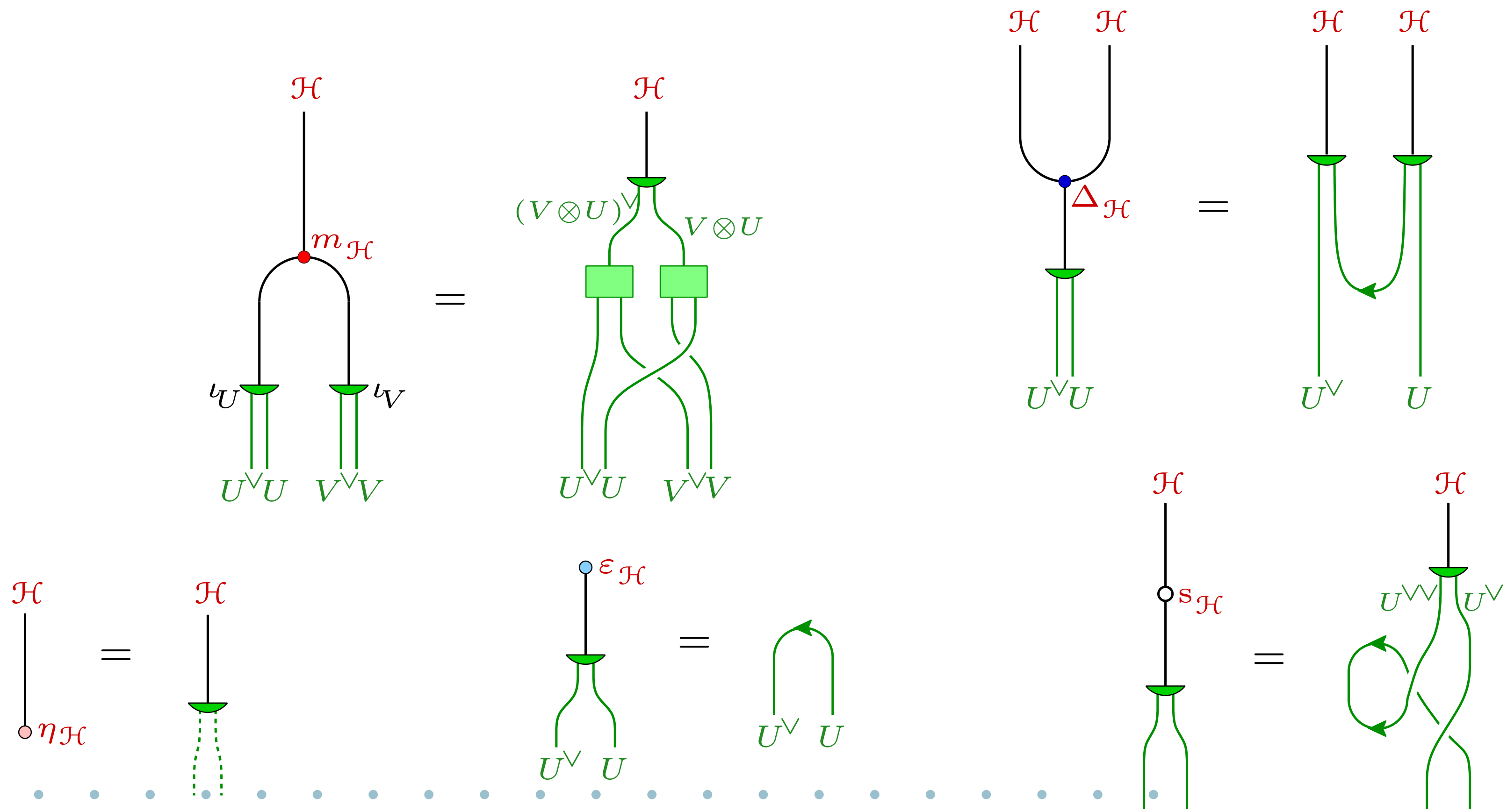

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Reminder IV : Representations of mapping class groups:

- Theorem : $\mathcal{D} = \mathcal{C} \equiv H$-mod$_{fd}$ $\Longrightarrow$ $\mathcal{H}$ is *modular* :
  - projective rep $\pi_{g,m}$ of Map_Σ on $\mathrm{Hom}_\mathcal{C}(\mathcal{H}^{\otimes g}, U_1 \otimes \cdots \otimes U_m)$
  - two-sided integral $\Lambda_\mathcal{H}$
  - non-degenerate Hopf pairing for $\mathcal{H}$

Goal: CFT correlators

Setting :

- Category of interest for CFT: $\overline{\mathcal{C}} \boxtimes \mathcal{C}$, for $\mathcal{C}$ a finite $\Bbbk$-linear ribbon category.
- For $\mathcal{C} = H\text{-mod}_{fd}$, with H a ribbon Hopf algebra, the category $\overline{\mathcal{C}} \boxtimes \mathcal{C}$ is the category of H-bimodules with a specific braiding
- Braided functor $\overline{\mathcal{C}} \boxtimes \mathcal{C} \to \mathcal{Z}(\mathcal{C})$ is equivalence for factorizable categories.
- The coend $K = \int^X X^\vee \otimes X$

  of $F_\otimes : (\overline{\mathcal{C}} \boxtimes \mathcal{C})^{\mathrm{op}} \times (\overline{\mathcal{C}} \boxtimes \mathcal{C}) \ni (X, Y) \longmapsto X^\vee \otimes Y \in \overline{\mathcal{C}} \boxtimes \mathcal{C}$

  exists and is naturally a modular Hopf algebra in $\overline{\mathcal{C}} \boxtimes \mathcal{C}$

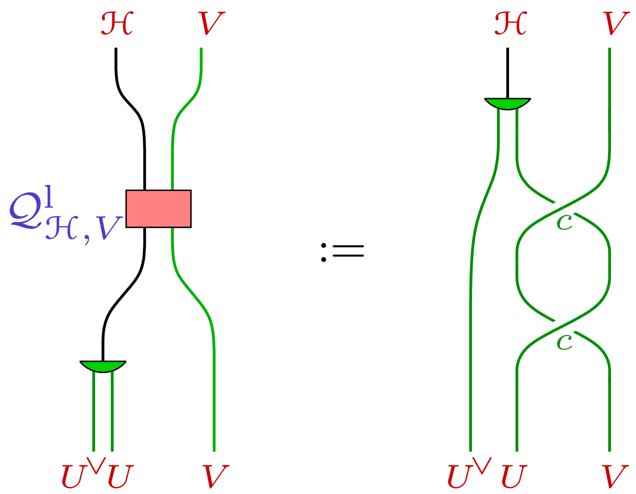
Motivation from full, local CFT: bulk fields are objects in $\overline{\mathcal{C}} \boxtimes \mathcal{C}$

Goal : Establish the following structure inspired by CFT:

- For any ribbon Hopf algebra automorphism $\omega : H \to H$
  A Frobenius algebra F_ω in $\overline{\mathcal{C}} \boxtimes \mathcal{C}$ with $\mathcal{C} = H\text{-mod}_{fd}$
- An vector $Cor^\omega_\Sigma \in \mathrm{Hom}_{\overline{\mathcal{C}} \boxtimes \mathcal{C}}(K^{\otimes g}, F_\omega^{\otimes m})$ that is invariant, $\pi_{g,m}(Cor^\omega_\Sigma) = Cor^\omega_\Sigma$

$\mathcal{H}$ Hopf algebra in braided tensor category $\mathcal{C}$ gives structure on all objects of $\mathcal{C}$:

- For every object $V \in \mathcal{C}$: partial monodromy $\mathcal{Q}^{\mathrm{l}}_{\mathcal{H},V} \in \mathrm{End}_{\mathcal{C}}(\mathcal{H} \otimes V)$:



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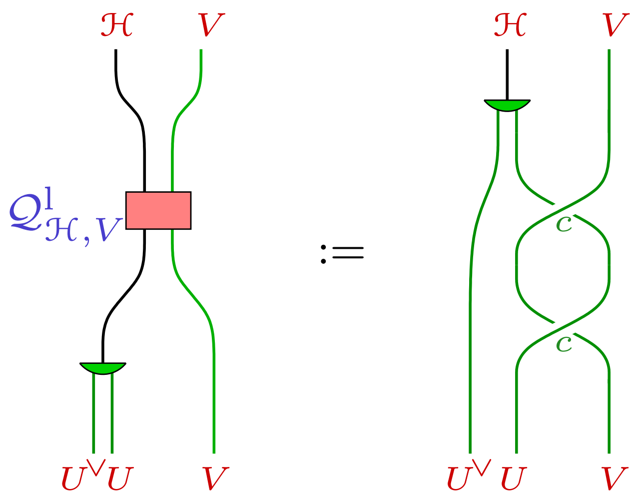

- Proposition: The partial monodromy $\mathcal{Q}^{\mathrm{l}}_{\mathcal{H},V}$ endows every object $V \in \mathcal{C}$ with the structure of an $\mathcal{H}$-module $(V, \rho_V^{\mathcal{H}})$:

$$\rho_V^{\mathcal{H}} = (\varepsilon_{\mathcal{H}} \otimes \mathrm{id}_V) \circ \mathcal{Q}^{\mathrm{l}}_{\mathcal{H},V}$$

(We even get a fully faithful embedding $\mathcal{C} \to {}_{\mathcal{H}}\mathcal{YD}^{\mathcal{H}}$.)

Main results

- Theorem 1 : Let ω be a ribbon automorphism of H-mod. Then the coend

$$F_\omega = \int^U U^\vee \boxtimes \omega(U) \in \overline{\mathcal{C}} \boxtimes \mathcal{C} \text{ of}$$

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \ni (U,V) \longmapsto U^\vee \boxtimes \omega(V)$$

exists and carries a natural structure of a commutative, cocommutative, symmetric Frobenius algebra in $\overline{\mathcal{C}} \boxtimes \mathcal{C}$ with trivial twist.

Interpretation of Frobenius algebra F_ω: bulk algebra “of automorphism type”.

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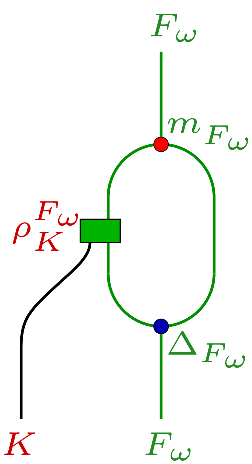
  Interpretation of Frobenius algebra F_ω: bulk algebra "of automorphism type".

- Theorem 2 : The following following family of morphisms is invariant under the action of the relevant mapping class group.
  Interpretation of ivnariants: CFT correlators

- Example:

invariant under the mapping class group of the two-punctured torus

$Cor^{\omega}_{1;1,1} :=$  $\in \mathrm{Hom}_{\overline{\mathcal{C}} \boxtimes \mathcal{C}}(K \otimes F_\omega, F_\omega)$

More examples of invariants

Example: 1+1 points on a torus

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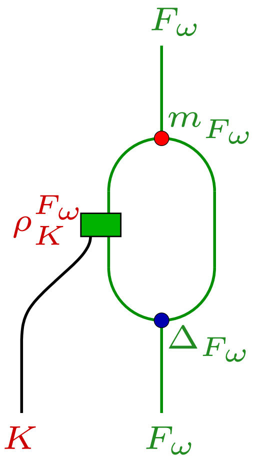

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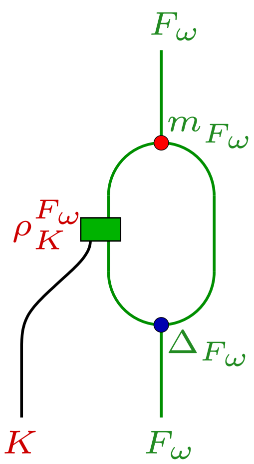

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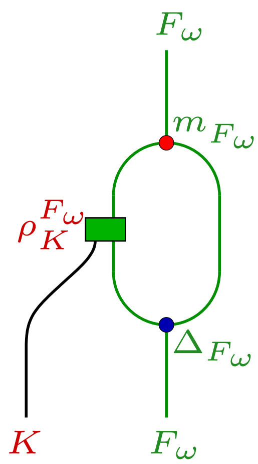

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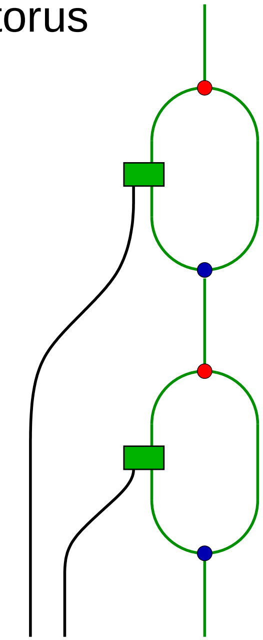
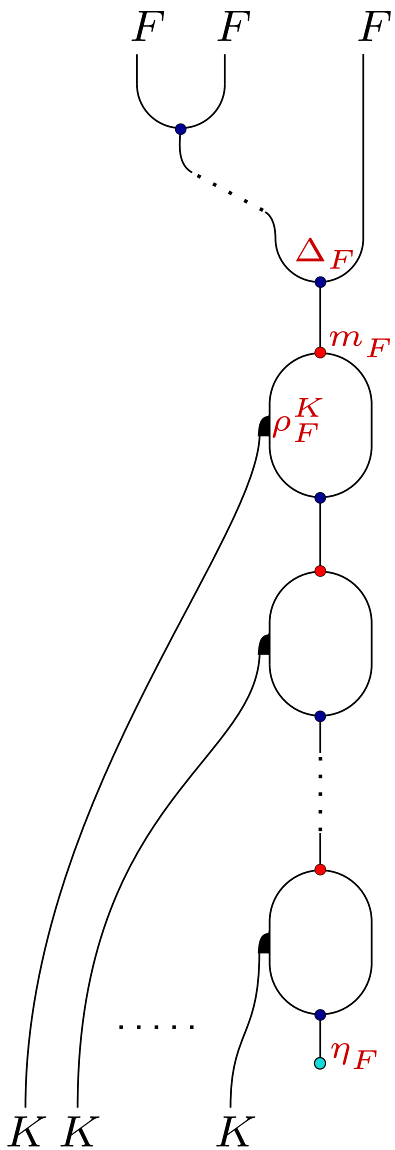
- Generalizes via product m_{F_ω} / coproduct Δ_{F_ω} to any number of incoming / outgoing marked points on a torus
- Generalizes by "composition of handles" to higher genus:

$Cor^{\omega}_{2;1,1} \;=$



The general form of the invariant

$$Cor^{\omega}_{g;0,n} \;=$$



Dream : generalizes to the relevant class of categories and obeys all factorization constraints.

The bulk Frobenius algebra

Restrict first to the identity automorphism: $\omega = \mathrm{id}$.

- Bulk Frobenius algebra $F := \int^X X^\vee \boxtimes X \in H$-bimod $\simeq \overline{H}$-mod $\boxtimes\, H$-mod :
  - $F = H^*$ with coregular left and right H-actions

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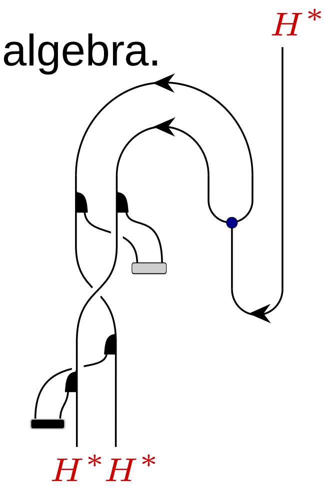
  - $F = H^*$ with coregular left and right H-actions
- Properties of the bulk Frobenius algebra :

  Commutative symmetric Frobenius algebra.
  - m_F commutative:

$m_F \circ c_{F,F} =$



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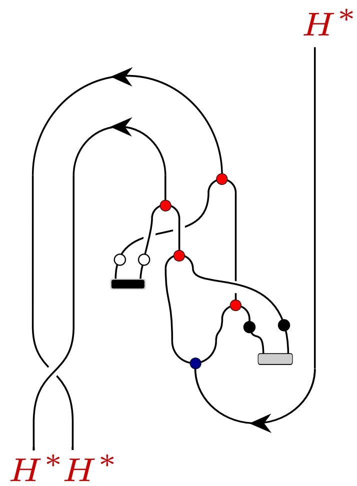

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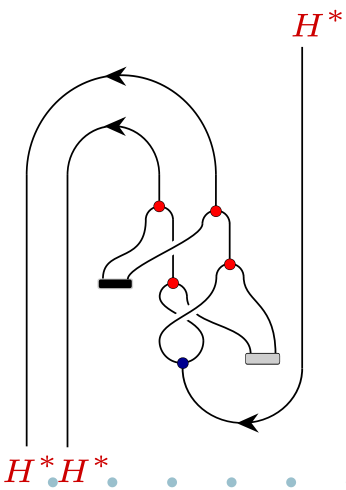

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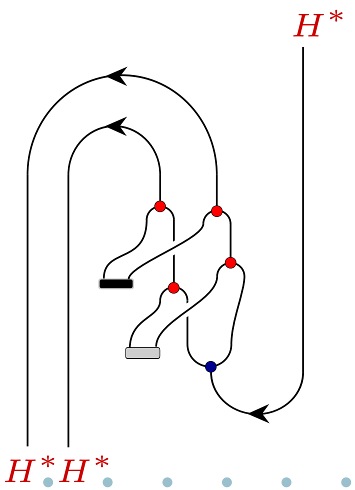

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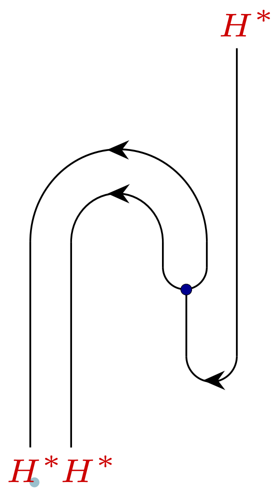
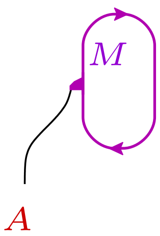


$= m_F$

The Cardy-Cartan invariant

- Character of a module M over an associative algebra A:

$\chi^A_M \; =$



$\in \mathrm{Hom}_{\mathcal{C}}(A, \mathbf{1})$

- Bulk handle Hopf algebra $K \cong \mathcal{H} \otimes \mathcal{H} \in \overline{\mathcal{C}} \boxtimes \mathcal{C}$.

- Partition function $Z_F := Cor_{1;0,0} = \chi^K_F \in \mathrm{Hom}_{\overline{\mathcal{C}} \boxtimes \mathcal{C}}(K, \mathbf{1})$ can be computed:

$$Z_F = \textstyle\sum_{i,j \in \mathcal{I}} c_{\bar{i},j} \; \chi^{\mathcal{H}}_i \otimes \chi^{\mathcal{H}}_j$$

with $c_{ij} = [P_j, S_i] = \dim_{\Bbbk} \mathrm{Hom}_H(P_i, P_j)$ the Cartan matrix of category H-mod :

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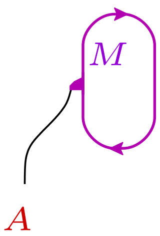

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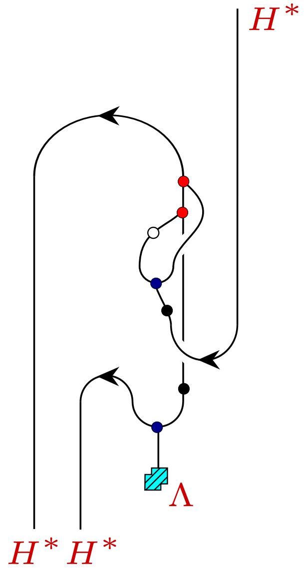
- Generalization :
  - ω Ribbon Hopf algebra automorphism of H
  - Coend: $F_\omega = \int^U U^\vee \boxtimes \omega(U)$

  $\Longrightarrow$ Bimodule $F_\omega := {}^{id}{}_{H}(F)^{\omega}$ with ω-twisted right action

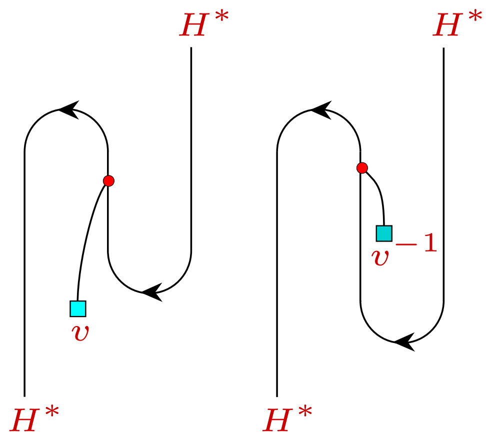
- F_ω carries a natural structure of a commutative, cocommutative, symmetric Frobenius algebra in H-mod$_{fd}$

- Invariant for $(0,1)$ -punctured torus :

$Cor_{1;0,1} \;=$

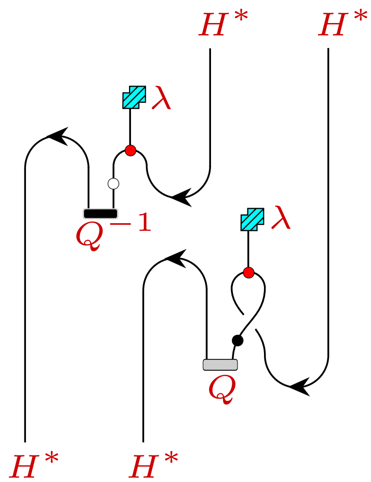


$\pi_{g,m}^{(K)}(T) \;=$



$$\pi_{g,m}^{(K)}(\Theta) = \theta_F = \mathrm{id}_F$$

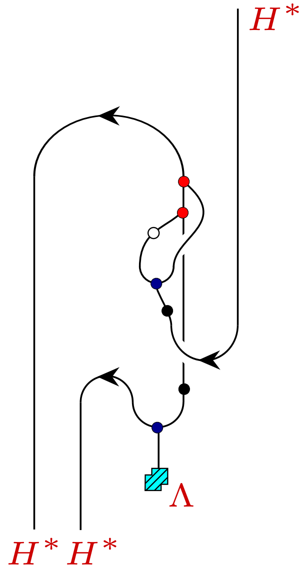
$\pi_{g,m}^{(K)}(S) \;=$

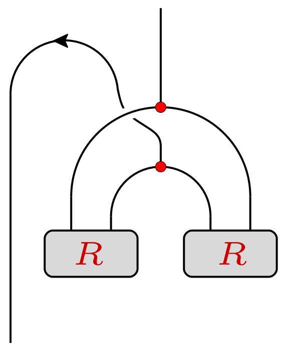


Invariance of genus-1 correlators

- Invariant for $(0,1)$ -punctured torus :

$$Cor_{1;0,1} \;=$$



- Drinfeld map : $f_Q \;:=\; (d_H \otimes \mathrm{id}_H) \circ (\mathrm{id}_{H^*} \otimes Q) \;=$  $\in \mathrm{Hom}(H^*, H)$

  - intertwines left coadjoint action and left adjoint action
  - $f_Q(\lambda) = \Lambda = f_{Q^{-1}}(\lambda)$ (with choice of scalars up to ± 1)

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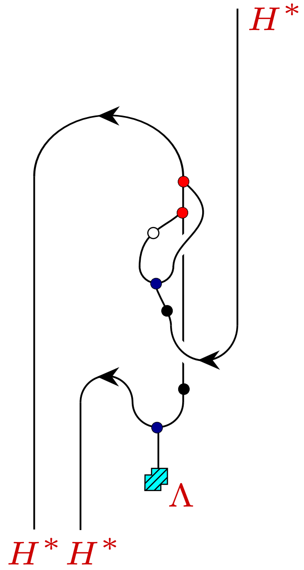
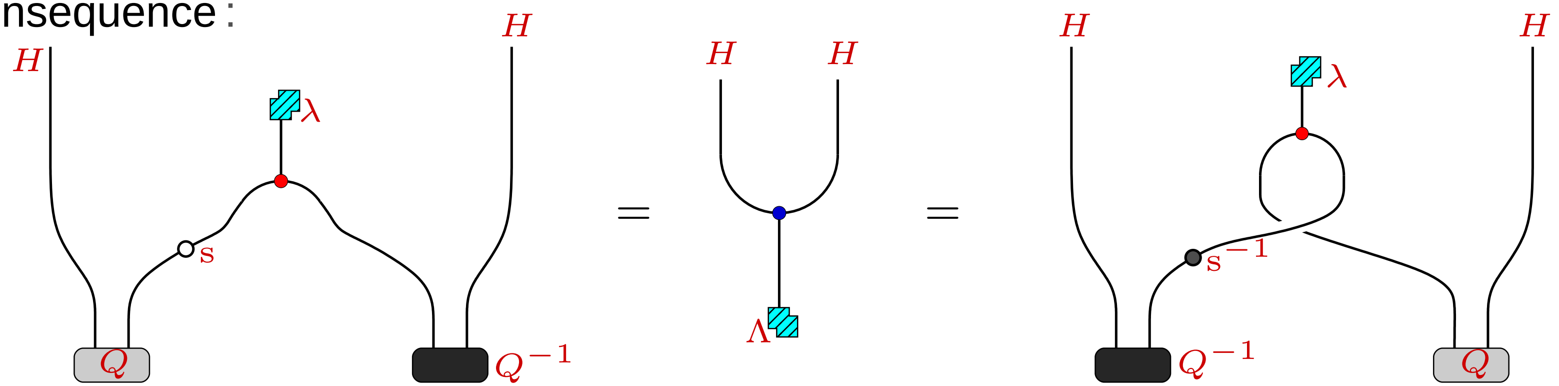


- Drinfeld map : $f_Q \,:=\, (d_H \otimes \mathrm{id}_H) \circ (\mathrm{id}_{H^*} \otimes Q)$
  - a consequence :



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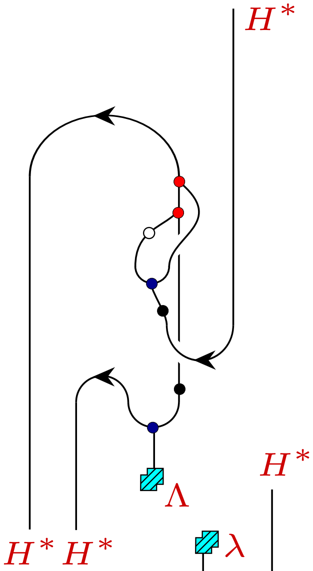
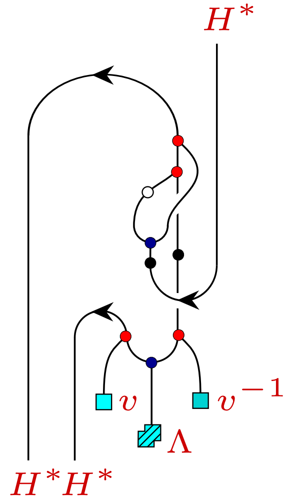


$Cor_{1;0,1} \circ \pi^{(K)}_{g,m}(T) =$



$= Cor_{1;0,1}$

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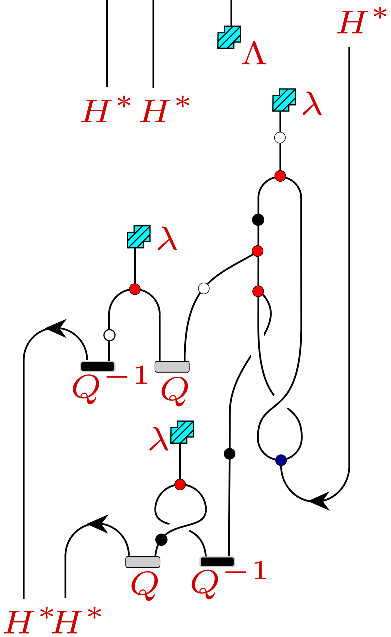

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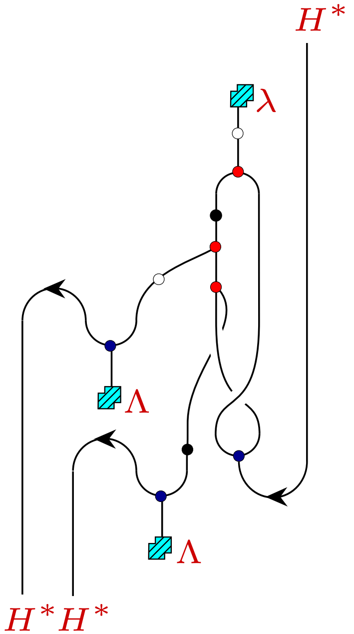

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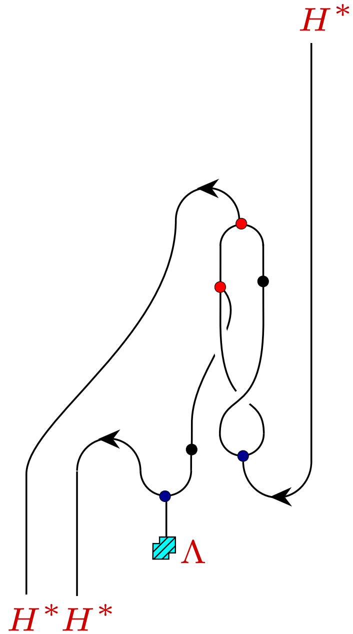

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