Differential forms in algebraic geometry — a new perspective in the singular case

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Today

1. Review of differential forms
2. Application: discrete invariants
3. Problems and solutions in the singular case

Differential forms

Back to calculus

- **derivative**

$$f:(a,b)\to\mathbb{R}\quad\Rightarrow\quad\frac{\partial f}{\partial x}:(a,b)\to\mathbb{R}$$

problem: depends on choice of coordinate!

- Better point of view: **differential form**

$$f\mapsto \mathrm{d}f=\frac{\partial f}{\partial x}\mathrm{d}x$$

- change of coordinate $f(x)=g(y(x))$

$$\frac{\partial f}{\partial x}\mathrm{d}x=\frac{\partial g}{\partial y}\frac{\partial y}{\partial x}\mathrm{d}x=\frac{\partial g}{\partial y}\mathrm{d}y$$

Back to calculus

M smooth manifold of dimension n
local coordinates $x_1, \dots, x_n$ near $P \in M$

Definition

$$f : M \to \mathbb{R} \quad \Rightarrow \quad \mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathrm{d}x_i$$

Geometric interpretation:
sections of the cotangent bundle T^*M.

Higher derivatives

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Definition

$$\Omega^q_M = \bigwedge^q \Omega^1_M$$

Differential graded algebra

- **product** $(\omega, \omega') \mapsto \omega \wedge \omega'$
- $\omega \wedge \omega' = (-1)^{\deg \omega \deg \omega'} \omega' \wedge \omega$
- **differential** $d : \Omega^q_M \to \Omega^{q+1}_M$
- $d \circ d = 0$
- $\omega \wedge \omega' \mapsto d\omega \wedge \omega' + (-1)^{\deg \omega} \omega \wedge d\omega'$
- in coordinates $f \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_q} \mapsto \sum_i \frac{\partial f}{\partial x_i} \mathrm{d}x_i \wedge \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_q}$

Algebraic differential forms

Observation

Derivatives of polynomials are polynomials

Consequence: Differential forms make sense in algebraic geometry

Algebraic varieties

k algebraically closed field, e.g. complex numbers $\mathbb{C}$

zero sets of polynomial equations in $\mathbb{A}^n = k^n$

Definition

1. An **affine variety** is given as

$$V(f_1, \ldots, f_m) = \{x \in k^n | f_i(x) = 0 \text{ for all } i\}$$

   for choice of $f_1, \ldots, f_m \in k[X_1, \ldots, X_n]$

2. **non-singular** if $\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j}$ has maximal rank, i.e., submanifold
3. General variety: locally affine, algebraic transition maps.

Differential forms for alg. varieties

$V = V(f_1, \dots, f_m)$ with $f_1, \dots, f_m \in k[X_1, \dots, X_n]$
$k[V] = k[X_1, \dots, X_n]/\langle f_1, \dots, f_m \rangle$

Definition

algebraic differential forms on V

- generators $dX_1, dX_2, \dots, dX_n$
- relations $df_1, \dots, df_m$
- $\Omega^1_V = \langle dX_1, \dots dX_n \rangle_{k[V]}/\langle df_1, \dots, df_m \rangle$
- Higher degree: $\Omega^q_V = \bigwedge^q \Omega^1_V$

Differential forms for alg. varieties

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Example: (affine plane) $\mathbb{A}^2 = k^2$, coordinates X, Y

- $\Omega^1_{\mathbb{A}^2} = \langle dX, dY \rangle_{k[X,Y]}$

Example: (hyperbola) $G = V(XY - 1) \cong k \smallsetminus \{0\}$
coordinates $X, Y = X^{-1}$

- $d(XY - 1) = YdX + XdY$
- $\Omega^1_G = \langle dX, dY \rangle_{k[X,X^{-1}]} / \langle YdX + XdY \rangle$
- $dY = -X^{-1} YdX = -X^{-2} dX \qquad$ in Ω^1_G
- $\Omega^1_G = k[X, X^{-1}]dX$

Use of differential forms

Classification of algebraic varieties!

- discrete invariants
- deformation theory
- period maps
- . . .

Application: source of discrete invariants

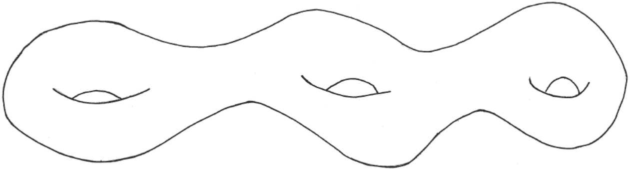
1. invariant: genus

Definition

genus of non-singular projective curve C

$$g = \dim_k \Omega^1_C(C)$$

Example: $k = \mathbb{C}$, $C = V(Y^4 - X^3 - X - 1)$, $g = 3$



2. invariant: de Rham cohomology

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X non-singular affine variety, characteristic 0
de Rham complex:

$$\Omega^0(X) \xrightarrow{d^0} \Omega^1(X) \xrightarrow{d^1} \Omega^2(X) \xrightarrow{d^2} \dots, \qquad d^i \circ d^{i-1} = 0$$

Definition

algebraic de Rham cohomology

$$H^i_{\mathrm{dR}}(X) = \mathrm{Ker}(d^i)/\mathrm{Im}(d^{i-1})$$

Example: $X = C$ projective curve

$$\dim_k H^1_{\mathrm{dR}}(X) = 2g$$

2. invariant: de Rham cohomology

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Theorem

$k = \mathbb{C}$, X non-singular variety over $\mathbb{C}$.

$$H^i_{\mathrm{dR}}(X) \cong H^i_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{C})$$

- Completely algebraic way of defining these invariants
- story continues with Hodge theory (Deligne)
- $k = \mathbb{Q}$: can be used to define period numbers ($\rightarrow$ number theory, mathematical physics)

2. invariant: de Rham cohomology

Example: $k = \mathbb{C}$, hyperbola $G = V(XY - 1)$

$$\Omega^*_G = \left[\mathbb{C}[X, X^{-1}] \xrightarrow{d} \mathbb{C}[X, X^{-1}]dX\right]$$

$$H^0_{\mathrm{dR}}(X) = \mathrm{Ker}\, d = \mathbb{C}$$

$$H^1_{\mathrm{dR}}(X) = \mathbb{C}[X, X^{-1}dX]/\mathrm{Im}\, d = \mathbb{C}\frac{dX}{X}$$

- $G = \mathbb{C} \setminus \{0\}$ homotopy equivalent to S^1

3. invariant: Kodaira dimension

- input: $\omega = \Omega^d_X$ for X non-singular variety of dimension d
- output: $\mathrm{Kd}(X) \in \{-\infty, 0, \dots, d\}$

Example: C non-singular projective curve

- $\mathrm{Kd}(X) = -\infty \Leftrightarrow g = 0$ (parabolic)
- $\mathrm{Kd}(X) = 0 \quad \Leftrightarrow g = 1$ (elliptic/flat)
- $\mathrm{Kd}(X) = 1 \quad \Leftrightarrow g \geq 2$ (hyperbolic)

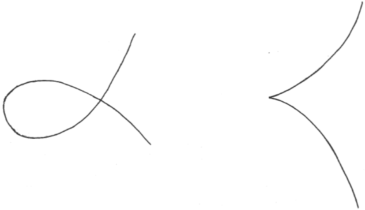
For algebraic geometers: $s_0, \dots, s_N$ basis of $\omega^{\otimes n}(X)$

$$\pi_n : X \longrightarrow \mathbb{P}^N,$$
$$x \mapsto [s_0(x) : \cdots : s_N(x)]$$

Definition: $\mathrm{Kd}(X) = \max_n \dim \pi_n(X)$

Problems and solutions in the singular case

Problems in the singular case

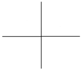


- Same definitions possible, give wrong answer
- Ω^1_V no longer a vector bundle

Remark

This is not surprise. A variety is defined to be non-singular if Ω^1_V is a vector bundle.

Example:



- $V = V(XY) \subset k^2$
- $\omega = XdY = -YdX \neq 0$
- vanishes on complement of $(0,0)$
- $dX \wedge dY \neq 0$

Ad hoc replacements in the singular case

Different replacements in use

- Torsion free differentials: $\Omega^q_X/\text{torsion}$
- Reflexive differentials: $\mathscr{O}_X$-double dual of Ω^q_X
- Dubois complex inspired by Hodge theory
- ...

All are useful for certain applications,
e.g., classification of algebraic varieties

New perspective in the singular case

- —, C. Jörder: Differential forms in the h-topology.
  Algebr. Geom. 1 (2014), no. 4, 449–478.
- —, S. Kebekus, S. Kelly: Differential forms in positive characteristic avoiding resolution of singularities,
  preprint 2014, arXiv:1407.5786

Key idea: Change the topology!

Grothendieck topologies

- Algebraic varieties are topological spaces.
- Very few open sets, e.g., all open sets dense in $\mathbb{A}^n$.
- Grothendieck ca. 1960: generalize the notion of topology
- use broader class of morphisms $V \to X$ in place of open subsets

Example: X topological manifold, allow all local homeomorphisms $V \to X$.

Extremely successful in algebraic geometry!

h-topology (Voevodsky 1996)

generated by

- open subsets
- $\tilde{X} \to X$ proper surjective (think: preimages of compact sets are compact)

Theorm (Hironaka 1964)

X variety of characteristic 0. Then there is $\tilde{X} \to X$ proper surjective with $\tilde{X}$ non-singular.

Consequence: locally in the h-topology, every variety is non-singular!

h-differentials

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h-locally given by algebraic differentials

Definition

Let Ω^q_h be the sheafification of $X \mapsto \Omega^q(X)$ in the h-topology

more concretely:

$$\Omega^q_h(X) = \mathrm{Ker}\left(\Omega^q(X_0) \xrightarrow{p_1^* - p_2^*} \Omega^q(X_1)\right)$$

- $X_0 \to X$ h-cover with X_0 non-singular,
- $X_1 \to X_0 \times_X X_0$ h-cover with X_1 non-singular

Results (characteristic 0) with C. Jörder

- nothing changes in the non-singular case, including cohomology
- if X has klt-singularities (mild singularities of minimal model program):
  h-differentitals equal reflexive differentials
- always related to complex of Dubois differentials
- define algebraic de Rham cohomology

Why I like them

Unifies ad hoc definitions, simplifies proofs,
very natural

Results (characteristic p) with S. Kelly, S. Kebekus

New problem: **Frobenius** $x \mapsto x^p$ has

$$dF(x) = dx^p = px^{p-1}dx = 0.$$

Consequence: $\Omega_h^q = 0$

Instead: cdh-topology or eh-topology

- differentials do not change in the non-singular case
- bad news: torsion exists, not functorial
- cohomology: work in progress,
  e.g. well-definedness of rational singularities
- ok under resolution of singularities

Earlier work

1. Geißer 2006
2. Lee 2009
3. Cortiñas, Haesemeyer, Schlichting, Walker and Weibel 2008-2013
4. Beilinson 2012

all concentrate on de Rham cohomology
homotopy invariant case $\to$ motives

Sales pitch

h-topology is very useful for individual Ω^q

Conclusion

- differential forms also work in algebraic geometry
- very useful source of invariants in the non-singular case
- h-differentials replace ad hoc definitions in the singular case
- work in progress in positive characteristic

Thank you